

On resolutions of diagrams of algebras

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July 8, 2011

Abstract

We prove a restricted version of a conjecture by M. Markl made in [7] on resolutions of an operad describing diagrams of algebras. We discuss a particular case related to the Gerstenhaber-Schack diagram cohomology.

1 Introduction

As explained in [9], the operadic cohomology gives a systematic way of constructing cohomology theories for algebras over an operad \mathcal{P} . The corresponding deformation complex carries an L_∞ -structure describing deformations of \mathcal{P} -algebras. To make this explicit, one has to find a free resolution of \mathcal{P} .

In particular, we can apply this to the coloured operad $\mathcal{A}_\mathcal{C}$ describing a \mathcal{C} -shaped diagram of \mathcal{A} -algebras. An important particular case is \mathcal{C} consisting of a single morphism. This is discussed in [7],[3] and also, indirectly, in the definition of (weak) A_∞ and L_∞ morphisms. More complicated categories \mathcal{C} received very little attention. In [7], M. Markl discussed examples leading to the notions of homotopy of \mathcal{A} -algebra morphisms and homotopy isomorphism of \mathcal{A} -algebras. In the end of the paper, a conjecture partially describing resolutions of $\mathcal{A}_\mathcal{C}$ for any \mathcal{A} and \mathcal{C} appears. In particular, it settles the question of the existence of the minimal resolution of $\mathcal{A}_\mathcal{C}$. We discuss this conjecture and prove it in the restricted case of \mathcal{A} being a Koszul operad with generating operations concentrated in a single arity and degree, see the main Theorem 3.15.

The idea is to glue together a *minimal* resolution of \mathcal{A} and any cofibrant free resolution of \mathcal{C} . The generators of the resulting resolution \mathcal{D}_∞ are described explicitly as well as the principal part of the differential ∂ . To state the theorem precisely requires some preliminary work.

First, we discuss operadic resolutions \mathcal{C}_∞ of categories. The operads in question are concentrated in arity 1, hence this is just a “coloured” version of classical homological algebra. We deal with maps $\llbracket - \rrbracket_n : \mathcal{C}_\infty \rightarrow \mathcal{C}_\infty^{\otimes n}$ with certain prescribed properties. These are needed to construct the principal part of ∂ . We show that these maps are

*The author was supported by GACR 201/09/H012 and by SVV-2011-263317.

induced by certain coproducts on \mathcal{C}_∞ , thus relating them to (coloured) dg bialgebra structures on \mathcal{C}_∞ .

The proof of the main theorem follows the ideas of M. Markl from [7]. It is necessarily more complicated technically and we discuss it in detail in a separate section. We find it convenient to recall some technical results of coloured operad theory, namely a version of the Künneth formula for the composition product \circ , which is very useful for homological computations. Hence we spend some time in the initial part of the paper explaining basics, though we expect the reader is already familiar with coloured operads.

The case \mathcal{C}_∞ being the bar-cobar resolution is particularly interesting. Here, \mathcal{C}_∞ has a topological flavour, it is completely explicit and we even make $\llbracket - \rrbracket$ explicit. The resulting resolution \mathcal{D}_∞ conjecturally gives rise to the Gerstenhaber-Schack complex for diagram cohomology [4].

Finally, let me thank Martin Markl for many useful discussions.

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In **Section 2**, we briefly recall basic notions of coloured operad theory. We focus on the interplay between the colours and Σ action. We prove a version of the Künneth formula in Section 2.2. It computes the homology of the composition product.

In **Section 3**, we prepare necessary notions to formulate the main theorem. In Section 3.1, we discuss operadic resolutions of categories and give several examples. In Section 3.2, we introduce $\llbracket - \rrbracket_n$ maps, certain combinatorial structures on the resolution of the category. We prove that these maps always exist and recall some examples from the literature. We show that $\llbracket - \rrbracket_n$'s are induced by $\llbracket - \rrbracket_2$, which is a certain coproduct on the resolution. In Section 3.3, we explain how diagrams of algebras are described by coloured operads and show that this construction is functorial and quism-preserving. Section 3.4 contains the statement of the main theorem and compares it to the conjecture by M. Markl.

In **Section 4**, the main theorem is proved. In Section 4.2, we try to explain the structure of the proof and to point out the places where an improvement might be possible.

In **Section 5**, we recall the bar-cobar resolution of the category, then we make $\llbracket - \rrbracket_n$'s explicit by endowing the resolution with a (coloured) bialgebra structure. Finally, we discuss the conjectural relation to Gerstenhaber-Schack diagram cohomology.

2 Basics

2.1 Conventions and reminder

We will use the following notations and conventions:

- \mathbb{N}_0 is the set of natural numbers including 0.
- k is a fixed field of characteristics 0.
- $k\langle S \rangle$ is the k -linear span of the set S .
- \otimes always means tensor product over k .
- Σ_n is the permutation group on n elements.
- V denotes a set (of colours¹).
- $\text{ar}(x)$ is arity of the object x , whatever x is.
- Vector spaces over k are called k -modules, chain complexes of vector spaces over k with differential of degree -1 are called dg - k -modules and morphisms of chain complexes are called just *maps*. Chain complexes are assumed **non-negatively graded** unless stated otherwise. The degree n summand of dg k -mod C is denoted C_n . We let $C_{\leq n} := \bigoplus_{0 \leq i \leq n} C_i$ and similarly for other inequality symbols. Similar notation is used e.g. for V - Σ -modules of Definition 2.2.
- $\uparrow C$ denotes the *suspension* of the graded object C , that is $(\uparrow C)_n = C_{n-1}$. Similarly, the *desuspension* is defined by $(\downarrow C)_n = C_{n+1}$.
- $|x|$ is the degree of an element x of a dg - k -module.
- $H_*(C)$ is homology of the object C , whatever C is.
- *Quism* is a map f of dg - k -modules such that the induced map $H_*(f)$ on homology is an isomorphism.

We extend the notation introduced in section Basics of [1] for V -coloured non- Σ operads to V -coloured Σ -operads.

2.1 Definition. A permutation $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ will also be denoted by $[\sigma(1)\sigma(2) \cdots \sigma(n)]$.

Let S be any set. Let $\vec{s} = (s_1, \dots, s_n) \in S^n$. If a context is clear, we may use this vector notation without explanation. S^n carries a right Σ_n action

$$\vec{s} \cdot \sigma := (s_{\sigma(1)}, \dots, s_{\sigma(n)}).$$

If $f : A^{\otimes n} \rightarrow A$ is a linear map, the right Σ_n action on f is defined by

$$(f \cdot \sigma)(\vec{a}) := f(\sigma \cdot \vec{a}) := f(\vec{a} \cdot \sigma^{-1}) = f(a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(n)})$$

for $\vec{a} \in A^n$. This is useful for intuitive understanding of the right Σ_n action on elements of an operad. While drawing pictures, we use the convention that into a leaf labelled i , the i^{th} input element is inserted. Hence element $a \cdot \sigma$ is drawn with labels $\sigma^{-1}(1), \sigma^{-1}(2), \dots, \sigma^{-1}(n)$ from left to right, e.g.

$$\left(\begin{array}{c} a \\ \diagup \quad | \quad \diagdown \\ 1 \quad 2 \quad 3 \end{array} \right) \cdot [312] = \begin{array}{c} a \\ \diagup \quad | \quad \diagdown \\ 2 \quad 3 \quad 1 \end{array} .$$

¹ V actually stands for Vertices, which will become apparent later.

For $\vec{v} \in V^n$, let

$$\Sigma_{\vec{v}} := \{\sigma \in \Sigma_n \mid \vec{v} = \vec{v} \cdot \sigma\} = \{\sigma \in \Sigma_n \mid v_i = v_{\sigma(i)} \text{ for each } 1 \leq i \leq n\}$$

be the stabilizer of \vec{v} under the action of Σ_n .

2.2 Definition. A **dg V - Σ -module** X is a set

$$\{X(n) \mid n \in \mathbb{N}_0\}$$

of dg right $k\langle \Sigma_n \rangle$ -modules such that each of them decomposes

$$X(n) = \bigoplus_{\substack{v \in V, \\ v_1, \dots, v_n \in V}} X \left(\begin{array}{c} v \\ v_1, \dots, v_n \end{array} \right)$$

as a dg k -module and $\sigma \in \Sigma_n$ acts by a dg k -module morphism

$$\cdot \sigma : X \left(\begin{array}{c} v \\ v_1, \dots, v_n \end{array} \right) \rightarrow X \left(\begin{array}{c} v \\ v_{\sigma(1)}, \dots, v_{\sigma(n)} \end{array} \right).$$

It follows that $X \left(\begin{array}{c} v \\ \vec{v} \end{array} \right)$ is a dg $k\langle \Sigma_{\vec{v}} \rangle$ -module. In particular, the differential commutes with the $k\langle \Sigma_{\vec{v}} \rangle$ action.

A **dg V - Σ -operad** is a dg V - Σ -module with the usual operadic compositions \circ_i . The axioms these compositions satisfy are the same as those for non- Σ dg V -operad (see [1], Definition 2.1) and we moreover require \circ_i 's to be equivariant in the usual sense (see [10], Definition 1.16 for noncoloured case).

We usually omit the prefix Σ and non- Σ . If a, b are elements of an operad \mathcal{A} and $\text{ar}(a) = 1$, we usually abbreviate $ab := a \circ b := a \circ_1 b$. If $\text{ar}(a) = n$, we also abbreviate $a(b_1 \otimes \dots \otimes b_n) := (\dots((a \circ_1 b_1) \circ_2 b_2) \dots) \circ_n b_n$. If V is a single element set, we omit the prefix “ V –”, otherwise we strictly keep the prefix.

Now we discuss the composition product \circ on the category of V - Σ -modules. We need some preliminary notions first.

2.3 Definition. Let l_1, \dots, l_m be nonnegative integers. For $n := l_1 + \dots + l_m$, there is the inclusion

$$\Sigma_{l_1} \times \dots \times \Sigma_{l_m} \hookrightarrow \Sigma_n$$

given by

$$(\lambda_1 \times \dots \times \lambda_m)(l_1 + \dots + l_{i-1} + j) := l_1 + \dots + l_{i-1} + \lambda_i(j),$$

where $1 \leq i \leq m$ and $1 \leq j \leq l_i$. If $l_i = 0$, we set $\Sigma_{l_i} = \Sigma_0 := \{1\}$.

Let $\tau \in \Sigma_m$. Denote

$$\begin{aligned} \overline{\tau} := [& l_1 + \dots + l_{\tau(1)-1} + 1, \dots, l_1 + \dots + l_{\tau(1)}, \\ & l_1 + \dots + l_{\tau(2)-1} + 1, \dots, l_1 + \dots + l_{\tau(2)}, \\ & \dots, \\ & l_1 + \dots + l_{\tau(m)-1} + 1, \dots, l_1 + \dots + l_{\tau(m)}]. \end{aligned}$$

If $l_i = 0$, the block $l_1 + \cdots + l_{\tau(i)-1} + 1, \dots, l_1 + \cdots + l_{\tau(i)}$ is empty and therefore is omitted in the expression above. Equivalently, the above formula states

$$\bar{\tau}(l_{\tau(1)} + l_{\tau(2)} + \cdots + l_{\tau(i-1)} + j) := l_1 + l_2 + \cdots + l_{\tau(i)-1} + j$$

for any $1 \leq i \leq m$ and $1 \leq j \leq l_i$. $\bar{\tau}$ is called (l_1, \dots, l_m) -**block permutation** corresponding to τ .

2.4 Example. • $[21] \times 1 \times [312] = [213645]$

- $(2, 1, 3)$ -block permutation corresponding to $\tau = [231]$ is $\overline{[231]} = [345612]$:

$$\begin{array}{cccccc} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} & \boxed{6} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \boxed{3} & \boxed{4} & \boxed{5} & \boxed{6} & \boxed{1} & \boxed{2} \end{array}$$

2.5 Definition. Fix $v \in V$ and $\vec{v} = (v_1, \dots, v_n) \in V^n$. Let $\mathcal{A} = (\mathcal{A}, \partial_{\mathcal{A}})$, $\mathcal{B} = (\mathcal{B}, \partial_{\mathcal{B}})$ be dg V - Σ -modules, let l_1, \dots, l_m be nonnegative integers such that $l_1 + \cdots + l_m = n$. For each $1 \leq i \leq m$, let $\vec{w}_i = (w_{i,1}, \dots, w_{i,l_i}) \in V^{l_i}$. Denote $\vec{W} = (\vec{w}_1, \dots, \vec{w}_m) = (w_{1,1}, \dots, w_{m,l_m}) \in V^n$. Let

$$\begin{aligned} \Sigma(\vec{W}, \vec{v}) &:= \left\{ \sigma \in \Sigma_n \mid \vec{W} \cdot \sigma = \vec{v} \right\} = \\ &= \left\{ \sigma \in \Sigma_n \mid w_{i,j} = v_{\sigma^{-1}(l_1 + \cdots + l_{i-1} + j)} \text{ for every } 1 \leq i \leq m, 1 \leq j \leq l_i \right\}. \end{aligned} \quad (1)$$

For fixed l_1, \dots, l_m and $\vec{w} = (w_1, \dots, w_m)$

$$\bigoplus_{\vec{W}} \mathcal{B} \binom{w_1}{\vec{w}_1} \otimes \cdots \otimes \mathcal{B} \binom{w_m}{\vec{w}_m} \otimes k \langle \Sigma(\vec{W}, \vec{v}) \rangle$$

is a dg² right $k \langle \Sigma_{l_1} \times \cdots \times \Sigma_{l_m} \rangle$ -module via

$$(b_1 \otimes \cdots \otimes b_m \otimes \sigma) \cdot (\lambda_1 \times \cdots \times \lambda_m) := (b_1 \cdot \lambda_1) \otimes \cdots \otimes (b_m \cdot \lambda_m) \otimes (\lambda_1 \times \cdots \times \lambda_m)^{-1} \sigma.$$

Denote the space of coinvariants of this $k \langle \Sigma_{l_1} \times \cdots \times \Sigma_{l_m} \rangle$ -module by the lower index $\Sigma_{l_1} \times \cdots \times \Sigma_{l_m}$.

Now assume only m is fixed and consider

$$\bigoplus_{\substack{l_1, \dots, l_m \\ \vec{w}}} \mathcal{A} \binom{v}{\vec{w}} \otimes \left(\bigoplus_{\vec{W}} \mathcal{B} \binom{w_1}{\vec{w}_1} \otimes \cdots \otimes \mathcal{B} \binom{w_m}{\vec{w}_m} \otimes k \langle \Sigma(\vec{W}, \vec{v}) \rangle \right)_{\Sigma_{l_1} \times \cdots \times \Sigma_{l_m}}.$$

This is dg right $k \langle \Sigma_m \rangle$ -module via

$$(a \otimes b_1 \otimes \cdots \otimes b_m \otimes \sigma) \cdot \tau = (a \cdot \tau) \otimes b_{\tau(1)} \otimes \cdots \otimes b_{\tau(m)} \otimes \overline{\tau^{-1}} \sigma,$$

where the bar denotes the corresponding (l_1, \dots, l_m) -block permutation of Definition 2.3. It is easy to verify that this action is well defined.

² $k \langle \Sigma(\vec{W}, \vec{v}) \rangle$ is concentrated in degree 0.

Finally, by taking the Σ_m coinvariants and summing over m in the above formula, we get the desired **composition product** of V - Σ -modules:

$$(\mathcal{A} \circ \mathcal{B}) \binom{v}{\vec{v}} := \bigoplus_m \left(\bigoplus_{l_1, \dots, l_m} \bigoplus_{\vec{w}} \mathcal{A} \binom{v}{\vec{w}} \otimes \left(\bigoplus_{\vec{W}} \mathcal{B} \binom{w_1}{\vec{w}_1} \otimes \dots \otimes \mathcal{B} \binom{w_m}{\vec{w}_m} \otimes k \langle \Sigma(\vec{W}, \vec{v}) \rangle \right)_{\Sigma_{l_1} \times \dots \times \Sigma_{l_m}} \right)_{\Sigma_m}, \quad (2)$$

where

- m runs through nonnegative integers,
- l_1, \dots, l_m run through nonnegative integers so that $l_1 + \dots + l_m = n$,
- $\vec{w} = (w_1, \dots, w_m)$ runs through V^m ,
- $\vec{W} = (\vec{w}_1, \dots, \vec{w}_m)$ runs through m -tuples of \vec{w}_i 's, where $\vec{w}_i \in V^{l_i}$,
- $\Sigma(\vec{W}, \vec{v})$ is given by (1).

To finish the definition of $\mathcal{A} \circ \mathcal{B}$, we let $\pi \in \Sigma_n$ act by

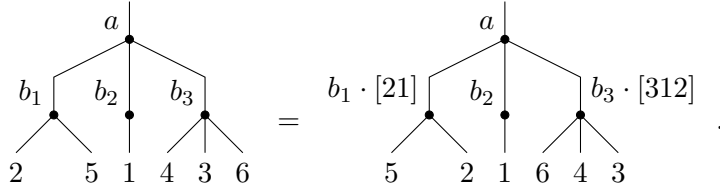
$$(a \otimes b_1 \otimes \dots \otimes b_m \otimes \sigma) \cdot \pi := a \otimes b_1 \otimes \dots \otimes b_m \otimes \sigma \pi.$$

We usually omit the coinvariants from the notation while dealing with elements of $\mathcal{A} \circ \mathcal{B}$.

The purpose of $\Sigma(\vec{W}, \vec{v})$ is to label the leaves so that for each i , the leaf labelled by i is of colour v_i . The purpose of the coinvariants is the usual one:

2.6 Example. By looking at the pictures, we find that we certainly want the equality

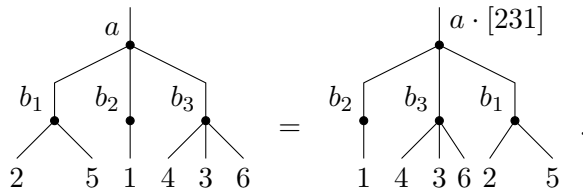
$$a \otimes b_1 \otimes b_2 \otimes b_3 \otimes [251436]^{-1} = a \otimes b_1[21] \otimes b_2 \otimes b_3[312] \otimes [521643]^{-1}$$



But since $[521643] = [251436]([21] \times 1 \times [312])$, the above equality is forced by taking the $\Sigma_{l_1} \times \dots \times \Sigma_{l_m}$ coinvariants.

We also want

$$a \otimes b_1 \otimes b_2 \otimes b_3 \otimes [251436]^{-1} = a[231] \otimes b_2 \otimes b_3 \otimes b_1 \otimes [143625]^{-1}$$



But $[143625] = [251436]\overline{[231]}$, hence this equality is forced by the Σ_m coinvariants.

2.7 Definition. Let X be a V - Σ -module. The free V -operad generated by X carries the *weight grading*

$$\mathbb{F}(X) = \bigoplus_{i \geq 0} \mathbb{F}^i(X),$$

where $\mathbb{F}^i(X)$ is spanned by free compositions of exactly i generators. If X is moreover dg V - Σ -module, the dg structure is inherited to $\mathbb{F}(X)$ in the obvious way and we obtain a free dg V -operad. However, $\mathbb{F}(X)$ can be equipped with a differential which doesn't come from X and in this case, $(\mathbb{F}(X), \partial)$ is called *quasi-free*.

Recall a quasi-free dg V -operad $(\mathbb{F}(X), \partial)$ is called **minimal** iff $\text{Im } \partial \subset \mathbb{F}^{\geq 2}(X)$. As usual, *free resolution* means a quism $(\mathbb{F}(X), \partial) \xrightarrow{\sim} (\mathcal{A}, \partial)$ with a quasi-free source. A *minimal resolution* is a resolution with a minimal source.

2.2 A Künneth formula

Our next task is to prove a version of the Künneth formula:

2.8 Lemma. Let $(\mathcal{A}, \partial_{\mathcal{A}})$, $(\mathcal{B}, \partial_{\mathcal{B}})$ be dg V - Σ -modules. Then there is a graded V - Σ -module isomorphism

$$H_*((\mathcal{A} \circ \mathcal{B}), \partial) \cong H_*(\mathcal{A}, \partial_{\mathcal{A}}) \circ H_*(\mathcal{B}, \partial_{\mathcal{B}}).$$

Proof. Let G be a finite group, let (M, ∂) be a dg $k\langle G \rangle$ -module. Obviously, ∂ descends to coinvariants, hence (M_G, ∂) is a dg $k\langle G \rangle$ -module too. We claim

$$H_*(M_G, \partial) \cong (H_*(M, \partial))_G. \quad (3)$$

By Maschke's theorem,

$$M = \bigoplus_{i \in I} M^i,$$

where M^i 's are irreducible $k\langle G \rangle$ -modules. ∂ is G -equivariant, hence for each i either $\partial M^i = 0$ or $\partial : M^i \xrightarrow{\cong} M^j$ is an isomorphism for some $j \neq i$. Denote

$$I_P := \{i \in I \mid \partial M^i = 0 \text{ and there is no } j \text{ such that } \partial M^j = M^i\}.$$

Also, for each i , either $\partial M_G^i = 0$ (iff $\partial M^i = 0$) or $\partial : M_G^i \xrightarrow{\cong} M_G^j$ is isomorphism for some $j \neq i$ (iff $\partial : M^i \xrightarrow{\cong} M^j$). Then

$$\begin{aligned} H_*(M_G, \partial) &= H_*\left(\bigoplus_{i \in I} M^i\right)_G, \partial) = H_*\left(\bigoplus_{i \in I} M_G^i, \partial\right) \cong \bigoplus_{i \in I_P} M_G^i \cong \\ &\cong \left(\bigoplus_{i \in I_P} M^i\right)_G = \left(H_*\left(\bigoplus_{i \in I} M^i, \partial\right)\right)_G \cong (H_*(M, \partial))_G. \end{aligned}$$

(3) is proved.

Let's set some shorthand notation. In (2), denote $B(\mathcal{B}) := \bigoplus_{\vec{w}} \mathcal{B}_{(\vec{w}_1)}^{(w_1)} \otimes \cdots \otimes \mathcal{B}_{(\vec{w}_m)}^{(w_m)} \otimes k\langle \Sigma(\vec{W}, \vec{v}) \rangle$. Denote $A(\mathcal{A}) := \mathcal{A}_{(\vec{w})}^{(v)}$ and $\Sigma := \Sigma_{l_1} \times \cdots \times \Sigma_{l_m}$. Notice that we are suppressing the dependency on l_1, \dots, l_m and \vec{w} . Omit v and \vec{v} too. Hence (2) becomes

$$\mathcal{A} \circ \mathcal{B} = \bigoplus_m \left[\bigoplus_{\substack{l_1, \dots, l_m \\ \vec{w}}} A(\mathcal{A}) \otimes B(\mathcal{B})_{\Sigma} \right]_{\Sigma_m}.$$

Let's compute:

$$\begin{aligned} H_*(\mathcal{A} \circ \mathcal{B}, \partial) &= \bigoplus_m H_*\left(\bigoplus_{\substack{l_1, \dots, l_m \\ \vec{w}}} A(\mathcal{A}) \otimes B(\mathcal{B})_\Sigma\right)_{\Sigma_m} \cong \bigoplus_m \left[\bigoplus_{\substack{l_1, \dots, l_m \\ \vec{w}}} H_*(A(\mathcal{A}) \otimes B(\mathcal{B})_\Sigma) \right]_{\Sigma_m} \cong \\ &\cong \bigoplus_m \left[\bigoplus_{\substack{l_1, \dots, l_m \\ \vec{w}}} H_*(A(\mathcal{A})) \otimes (H_*(B(\mathcal{B})))_\Sigma \right]_{\Sigma_m} \cong \dots \end{aligned}$$

The last isomorphism is provided by the usual Künneth formula and (3). Now trivially $H_*(A(\mathcal{A})) = A(H_*(\mathcal{A}, \partial_{\mathcal{A}}))$ and another application of the Künneth formula gives $H_*(B(\mathcal{B})) \cong B(H_*(\mathcal{B}, \partial_{\mathcal{B}}))$ and we finish:

$$\dots \cong \bigoplus_m \left[\bigoplus_{\substack{l_1, \dots, l_m \\ \vec{w}}} A(H_*(\mathcal{A}, \partial_{\mathcal{A}})) \otimes B(H_*(\mathcal{B}, \partial_{\mathcal{B}}))_\Sigma \right]_{\Sigma_m} = H_*(\mathcal{A}, \partial_{\mathcal{A}}) \circ H_*(\mathcal{B}, \partial_{\mathcal{B}}).$$

□

3 Statement of main theorem

3.1 Operadic resolution of category

Let \mathbf{C} be a small category and denote

$$V := \text{Ob } \mathbf{C}$$

the set of its objects. For a morphism $f \in \text{Mor } \mathbf{C}$, let $I(f)$ be its source (Input) and $O(f)$ its target (Output). Let \mathcal{C} be the operadic version of \mathbf{C} , that is

$$\mathcal{C} := k\langle \text{Mor } \mathbf{C} \rangle \tag{4}$$

is seen as a coloured V -operad concentrated in arity 1, where each $f \in \text{Mor } \mathbf{C}$ is an element of $\mathcal{C}_{I(f)}^{O(f)}$ and the operadic composition is induced by the categorical composition. Obviously, \mathcal{C} can be presented as

$$\mathcal{C} = \frac{\mathbb{F}(k\langle \text{Mor } \mathbf{C} - \{\text{identities}\} \rangle)}{(\text{relations})},$$

where each relator is generated by those of the form $r_1 - r_2$ with r_1, r_2 being operadic compositions of elements of $\text{Mor } \mathbf{C}$. Recall that the elements corresponding to the identities become a part of the free operad construction.

Every such V -operad \mathcal{C} has a free resolution of the form

$$\mathcal{C}_\infty := (\mathbb{F}(F), \partial) \xrightarrow{\sim} (\mathcal{C}, 0),$$

where the graded V - Σ -module³ $F = \bigoplus_{i \geq 0} F_i$ satisfies

Assumptions 3.1.

1. $F_0 = k\langle M \rangle$ for some $M \subset \text{Mor } \mathbf{C} - \{\text{identities}\}$,

³Of course, the action of Σ_1 carries no information and can be omitted.

2. $F_1 = k\langle R \rangle$, where for each $r \in R$, $\partial r = r_1 - r_2$ for some free operadic compositions r_1, r_2 of elements of $M \cup \{\text{identities}\}$.

The existence of such a resolution is quite obvious and we will give several examples below. A general example is given by the bar-cobar resolution, which will be discussed later in Section 5 in detail. Before giving the examples, we note that

$$\mathcal{C} \cong \frac{\mathbb{F}(F_0)}{(\partial F_1)}. \quad (5)$$

3.2 Example. Let \mathcal{C} be the category generated by 2 distinct morphisms between 3 distinct objects as in the picture:

$$V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$$

Then $\text{Ob } \mathcal{C} = V = \{V_1, V_2, V_3\}$, $\text{Mor } \mathcal{C} = \{1_{V_1}, 1_{V_2}, 1_{V_3}, f, g, h := gf\}$. The composition is obvious. The V -operad \mathcal{C} has colour decomposition $\mathcal{C}_{V_1}^{(V_2)} = k\langle f \rangle$, $\mathcal{C}_{V_2}^{(V_3)} = k\langle g \rangle$, $\mathcal{C}_{V_1}^{(V_3)} = k\langle h \rangle$. \mathcal{C} has the following 2 obvious resolutions:

1. Directly from the obvious presentation of \mathcal{C} , we get

$$(\mathbb{F}(k\langle f, g, h, H \rangle), \partial) \xrightarrow{\sim} (\mathcal{C}, 0),$$

where f, g, h are copies of the corresponding generators of \mathcal{C} and $I(H) = V_1$, $O(H) = V_3$. The degrees are as follows : $|f| = |g| = |h| = 0$ and $|H| = 1$. The differential ∂ vanishes on f, g, h and $\partial H = gf - h$.

2. A “smaller” resolution of \mathcal{C} is

$$(\mathbb{F}(k\langle f, g \rangle), 0) \xrightarrow{\sim} (\mathcal{C}, 0).$$

It has less generators because the existence of h is already forced by the existence of f, g . This is an example of a minimal resolution of Definition 2.7.

3.3 Example. The category

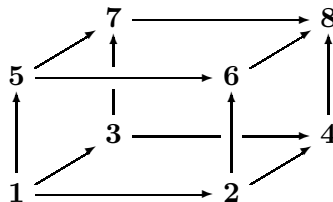
$$V_1 \xrightarrow{f} V_2$$

has, apart from the obvious one, a free resolution

$$\mathcal{C}_\infty := (\mathbb{F}(k\langle f, g, H \rangle), \partial) \xrightarrow{\sim} (\mathcal{C}, 0),$$

where $I(g) = I(H) = V_1$, $O(g) = O(H) = V_2$, $|g| = 0$, $|H| = 1$ and $\partial H = f - g$. It was observed in [7] that every algebra over \mathcal{C}_∞ corresponds to a pair of dg k -modules, a pair of morphisms f, g between these and a homotopy H between f and g . Hence even resolutions of boring categories, such as \mathcal{C} in this example, may lead to interesting concepts.

3.4 Example. Probably the simplest example of \mathcal{C} which can't be resolved in degrees 0 and 1 only is given by the commutative cube:



Objects (i.e. vertices) are denoted $\mathbf{1}, \dots, \mathbf{8}$, edges (and the corresponding generators of the resolution below) are denoted (ab) with $\mathbf{8} \geq a > b \geq \mathbf{1}$. The faces are denoted $(abcd)$ with $\mathbf{8} \geq a > b > c > d \geq \mathbf{1}$. Then

$$(\mathbb{F}\langle(\mathbf{21}), \dots, (\mathbf{4321}), \dots, H\rangle, \partial) \xrightarrow{\sim} (\mathcal{C}, 0)$$

is generated by all edges and faces and H so that the edges are of degree 0 and $I((ab)) = b$, $O((ab)) = a$; faces are of degree 1 and $I((abcd)) = d$, $O((abcd)) = a$; finally $|H| = 2$ and $I(H) = \mathbf{1}$, $O(H) = \mathbf{8}$. The differential is given by

$$\begin{aligned} \partial(ab) &= 0, \\ \partial(abcd) &= (ac)(cd) - (ab)(bd), \\ \partial H &= (\mathbf{84})(\mathbf{4321}) + (\mathbf{8743})(\mathbf{31}) - (\mathbf{8642})(\mathbf{21}) + \\ &\quad + (\mathbf{87})(\mathbf{7531}) - (\mathbf{8765})(\mathbf{51}) - (\mathbf{86})(\mathbf{6521}). \end{aligned}$$

The resolving morphism maps edges to edges and all other generators to 0. It is easy to verify that this is a minimal resolution.

We let the reader convince himself that \mathcal{C} can't indeed be resolved just in degrees 0 and 1. Rigorously, this would follow from the uniqueness of the minimal resolution together with a theorem asserting that any free resolution decomposes into a free product of a minimal resolution and an acyclic dg V -operad⁴. These theorems however go beyond the scope of this paper.

3.5 Example. An explicit resolution of the category generated by

$$V_1 \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} V_2$$

with relations

$$fg - 1_{V_2}, \quad gf - 1_{V_1}$$

was found in [8]. It contains a generator of *each* nonnegative degree.

3.2 $\llbracket - \rrbracket_n$ maps

Since \mathcal{C}_∞ is concentrated in arity 1, we won't distinguish between \mathcal{C}_∞ and $\mathcal{C}_\infty(1)$. Also observe, that V -operad concentrated in arity 1 is just a coloured⁵ dg associative algebra.

Consider the usual dg structure on $\mathcal{C}_\infty^{\otimes n}$. There is also a right action of Σ_n generated by transpositions as follows. Let $\tau \in \Sigma_n$ exchange i and j . Then

$$\begin{aligned} (r_1 \otimes \dots \otimes r_i \otimes \dots \otimes r_j \otimes \dots \otimes r_n) \cdot \tau &:= \\ = (-1)^{|r_i||r_j| + (|r_i| + |r_j|) \sum_{i < k < j} |r_k|} r_1 \otimes \dots \otimes r_j \otimes \dots \otimes r_i \otimes \dots \otimes r_n \end{aligned}$$

for any $r_1, \dots, r_n \in \mathcal{C}_\infty$ such that $I(r_i) = O(s_i)$ for all $1 \leq i \leq n$. Further, there is the factorwise composition on $\mathcal{C}_\infty^{\otimes n}$:

$$(r_1 \otimes \dots \otimes r_n) \circ (s_1 \otimes \dots \otimes s_n) := (-1)^{\sum_{n \geq i > j \geq 1} |r_i||s_j|} (r_1 s_1) \otimes \dots \otimes (r_n s_n). \quad (6)$$

⁴An analogue exists in rational homotopy theory - see [2], Theorem 14.9.

⁵The operations are defined only partially, respecting the colours.

It is easily seen that ∂ is a degree -1 derivation with respect to \circ :

$$\partial(R \circ S) = (\partial R) \circ S + (-1)^{|R|} R \circ \partial S$$

for any $R, S \in \mathcal{C}_\infty^{\otimes n}$. Also, \circ is Σ_n equivariant:

$$(R \circ S) \cdot \tau = (R \cdot \tau) \circ (S \cdot \tau).$$

The following lemma is a straightforward generalization of Definition 23 of [7]:

3.6 Lemma. For every integer $n \geq 1$, there is a linear map

$$\llbracket - \rrbracket_n : \mathcal{C}_\infty \rightarrow \mathcal{C}_\infty^{\otimes n}$$

satisfying for every $r, r' \in \mathcal{C}_\infty$

- (C1) $\llbracket r \rrbracket_n$ is Σ_n -stable,
- (C2) $\llbracket r \rrbracket_n \in \mathcal{C}_\infty^{(O(r))}_{I(r)}^{\otimes n}$,
- (C3) $\deg \llbracket r \rrbracket_n = \deg r$,
- (C4) $\llbracket f \rrbracket_n = f^{\otimes n}$ for every morphism $f \in M \subset F_0$ (recall 3.1),
- (C5) $\llbracket r \circ r' \rrbracket_n = \llbracket r \rrbracket_n \circ \llbracket r' \rrbracket_n$,
- (C6) $\partial \llbracket r \rrbracket_n = \llbracket \partial r \rrbracket_n$.

Proof. Fix n . We proceed by induction on degree d . (C4) defines $\llbracket - \rrbracket_n$ for M , we extend linearly to F_0 and then extend by (C5) to all of $\mathbb{F}(F_0)$. Obviously, (C1)–(C6) hold for $r, r' \in \mathbb{F}(F_0)$. Assume we have already defined $\llbracket - \rrbracket_n$ on $\mathbb{F}(F_{<d})$ so that (C2)–(C6) hold.

1. Let $d = 1$. By the assumptions 3.1, $F_1 = k\langle R \rangle$ and $f \in R$. We have $\partial f = r_1 - r_2$ as in 3.1, hence $\llbracket \partial f \rrbracket_n = r_1^{\otimes n} - r_2^{\otimes n}$. Define

$$\llbracket f \rrbracket_n^{\text{NS}} := \sum_{i=0}^{n-1} r_1^{\otimes i} \otimes f \otimes r_2^{\otimes n-i-1}. \quad (7)$$

An easy computation shows $\llbracket f \rrbracket_n^{\text{NS}}$ is a degree 1 element of $\mathcal{C}_\infty^{\otimes n}(O(f))_{I(f)}$ satisfying $\partial \llbracket f \rrbracket_n^{\text{NS}} = \llbracket \partial f \rrbracket_n$.

2. Let $d \geq 2$. Since $|\partial f| < d$, $\llbracket \partial f \rrbracket_n$ is already constructed and we are solving the equation

$$\partial \llbracket f \rrbracket_n = \llbracket \partial f \rrbracket_n$$

for an unknown $\llbracket f \rrbracket_n$ in the standard way. By the induction assumption, $\partial \llbracket \partial f \rrbracket_n = \llbracket \partial^2 f \rrbracket_n = 0$. By the usual Künneth formula, $\mathcal{C}_\infty^{\otimes n}$ is acyclic in positive degrees. Since $|\llbracket \partial f \rrbracket_n| = d-1 > 0$, we obtain a degree d element $\llbracket f \rrbracket_n^{\text{NS}} \in \mathcal{C}_\infty^{(O(f))}_{I(f)}^{\otimes n}$ such that $\partial \llbracket f \rrbracket_n^{\text{NS}} = \llbracket \partial f \rrbracket_n$.

Making $\llbracket f \rrbracket_n^{\text{NS}}$ to satisfy (C1) in characteristics 0 is easy:

$$\llbracket f \rrbracket_n := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \llbracket f \rrbracket_n^{\text{NS}} \cdot \sigma. \quad (8)$$

We now have $\llbracket f \rrbracket_n$ satisfying (C1)–(C4) and (C6) for every $f \in F_d$. Extend this to $\mathbb{F}(F_{\leq d})$ by (C5). By Σ_n equivariance of \circ , (C1) holds on $\mathbb{F}(F_{\leq d})$. Verifying (C2) and (C3) is trivial, hence it remains to check (C6). Let $f_1, \dots, f_m \in F_{\leq d}$:

$$\begin{aligned} \partial \llbracket f_1 \cdots f_m \rrbracket_n &= \partial (\llbracket f_1 \rrbracket_n \circ \cdots \circ \llbracket f_m \rrbracket_n) = \sum_{i=1}^m (-1)^{\epsilon_i} \llbracket f_1 \rrbracket_n \circ \cdots \circ \partial \llbracket f_i \rrbracket_n \circ \cdots \circ \llbracket f_m \rrbracket_n = \\ &= \sum_{i=1}^m (-1)^{\epsilon_i} \llbracket f_1 \rrbracket_n \circ \cdots \circ \llbracket \partial f_i \rrbracket_n \circ \cdots \circ \llbracket f_m \rrbracket_n = \\ &= \llbracket \sum_{i=1}^m (-1)^{\epsilon_i} f_1 \cdots \partial f_i \cdots f_m \rrbracket_n = \llbracket \partial(f_1 \cdots f_m) \rrbracket_n, \end{aligned}$$

where $\epsilon_i := |f_1| + \cdots + |f_{i-1}|$. Hence (C6) is valid for all elements of $\mathbb{F}(F_{\leq d})$ and the induction is finished. \square

3.7 Example. If \mathcal{C}_∞ is concentrated in degrees ≤ 1 , then we have explicit formulas (7) and (8) for $\llbracket - \rrbracket$ given in the proof.

3.8 Example. For the resolution of Example 3.5, the construction of $\llbracket - \rrbracket_n$'s using Lemma 3.6 is not explicit. In this case, $\llbracket - \rrbracket$ was found explicitly in [7], Remark 25.

The following lemma shows that $\llbracket - \rrbracket_2$ induces $\llbracket - \rrbracket_n$ for all $n \geq 3$. $\llbracket - \rrbracket_2$ can be thought of as a coproduct on \mathcal{C}_∞ . If $\llbracket - \rrbracket_2$ is moreover coassociative, then (C2),(C5) and (C6) means that $(\mathcal{C}_\infty, \circ, \llbracket - \rrbracket_2)$ is a coloured dg *bialgebra*.

3.9 Lemma. Let $\llbracket - \rrbracket_2^{\text{NS}} : \mathcal{C}_\infty \rightarrow \mathcal{C}_\infty \otimes \mathcal{C}_\infty$ be a linear map satisfying the conditions (C2)–(C6) of Lemma 3.6. Set

$$\llbracket - \rrbracket_n^{\text{NS}} := (\llbracket - \rrbracket_2^{\text{NS}} \otimes 1^{\otimes n-2})(\llbracket - \rrbracket_2^{\text{NS}} \otimes 1^{\otimes n-3}) \cdots (\llbracket - \rrbracket_2^{\text{NS}} \otimes 1) \llbracket - \rrbracket_2^{\text{NS}}. \quad (9)$$

Then for each $n \geq 3$, $\llbracket - \rrbracket_n^{\text{NS}}$ satisfies (C2)–(C6) and $\llbracket - \rrbracket_n$ defined by (8) satisfies (C1)–(C6). If $\llbracket - \rrbracket_2^{\text{NS}}$ is coassociative, i.e. $(\llbracket - \rrbracket_2^{\text{NS}} \otimes 1) \llbracket - \rrbracket_2^{\text{NS}} = (1 \otimes \llbracket - \rrbracket_2^{\text{NS}}) \llbracket - \rrbracket_2^{\text{NS}}$, then

$$(1^{\otimes i} \otimes \llbracket - \rrbracket_a^{\text{NS}} \otimes 1^{\otimes b-i-1}) \llbracket - \rrbracket_b^{\text{NS}} = \llbracket - \rrbracket_{a+b-1}^{\text{NS}}$$

for every $a, b \geq 2$, $0 \leq i \leq b-1$.

Proof. Conditions (C2)–(C4) for $\llbracket - \rrbracket_n^{\text{NS}}$ are easily seen to be satisfied.

We sketch a proof of (C5) by the standard flow diagrams. Let $\llbracket - \rrbracket_2^{\text{NS}}$ be represented

by $\begin{array}{c} \diagup \\ \diagdown \end{array}$, then $\llbracket - \rrbracket_n^{\text{NS}}$ is represented by $\overbrace{\begin{array}{c} \diagup \\ \vdots \\ \diagdown \end{array}}^n$. Let $\circ : \mathcal{C}_\infty \otimes \mathcal{C}_\infty \rightarrow \mathcal{C}_\infty$ of (6) be represented by $\begin{array}{c} \diagup \\ \diagdown \end{array}$, then $\circ : \mathcal{C}_\infty^{\otimes n} \otimes \mathcal{C}_\infty^{\otimes n} \rightarrow \mathcal{C}_\infty^{\otimes n}$ is represented, e.g. for $n = 3$, by $\begin{array}{c} \diagup \\ \diagdown \end{array}$. Observe that the signs are handled by the Koszul sign convention. The property (C5) for $\llbracket - \rrbracket_2^{\text{NS}}$ states

$$\begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array}. \quad (10)$$

For $n = 3$, we have to prove $\llbracket a \circ b \rrbracket_3^{\text{NS}} = \llbracket a \rrbracket_3^{\text{NS}} \circ \llbracket b \rrbracket_3^{\text{NS}}$, i.e.

$$\begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array}.$$

Applying (10) to the bold subgraph, we obtain



and another application of (10) on the bold subgraph gives the left hand side of the desired equality. The general case is analogous.

We prove (C6) by induction on n . $n = 2$ is the hypothesis. Let (C6) be true for $n - 1$ and let's compute:

$$\begin{aligned}
\partial \llbracket - \rrbracket_n^{\text{NS}} &= \partial(\llbracket - \rrbracket_2^{\text{NS}} \otimes 1^{\otimes n-2}) \cdots (\llbracket - \rrbracket_2^{\text{NS}} \otimes 1) \llbracket - \rrbracket_2^{\text{NS}} = \\
&= (\partial \llbracket - \rrbracket_2^{\text{NS}} \otimes 1^{\otimes n-2}) \llbracket - \rrbracket_{n-1}^{\text{NS}} + \sum_{i=0}^{n-3} (\llbracket - \rrbracket_2^{\text{NS}} \otimes 1^{\otimes i} \otimes \partial \otimes 1^{\otimes n-3-i}) \llbracket - \rrbracket_{n-1}^{\text{NS}} = \\
&= (\llbracket - \rrbracket_2^{\text{NS}} \otimes 1^{\otimes n-2}) (\partial \otimes 1^{\otimes n-2}) \llbracket - \rrbracket_{n-1}^{\text{NS}} + \\
&\quad + \sum_{i=0}^{n-3} (\llbracket - \rrbracket_2^{\text{NS}} \otimes 1^{\otimes n-2}) (1^{\otimes i+1} \otimes \partial \otimes 1^{\otimes n-3-i}) \llbracket - \rrbracket_{n-1}^{\text{NS}} = \\
&= (\llbracket - \rrbracket_2^{\text{NS}} \otimes 1^{\otimes n-2}) \partial \llbracket - \rrbracket_{n-1}^{\text{NS}} = (\llbracket - \rrbracket_2^{\text{NS}} \otimes 1^{\otimes n-2}) \llbracket - \rrbracket_{n-1}^{\text{NS}} \partial = \llbracket - \rrbracket_n^{\text{NS}} \partial.
\end{aligned}$$

The proof of the coassociativity statement is easy and we leave it to the reader. \square

Later, in Theorem 5.1, we will construct $\llbracket - \rrbracket$ on the bar-cobar resolution ΩBC of \mathcal{C} using this lemma out of a coassociative coproduct on ΩBC .

3.10 Example. Our assumptions 3.1 are important. Consider the category generated by a single morphism between two distinct objects as in Example 3.3. Then \mathcal{C} has yet another resolution: Take the same generators as in 3.3,

$$\mathcal{C}_\infty := (\mathbb{F}(k\langle f, g, H \rangle), \partial) \xrightarrow{\sim} (\mathcal{C}, 0),$$

but let

$$\partial H = f + g.$$

Then an elementary linear algebra shows that $f^{\otimes 2} + g^{\otimes 2}$ is a cycle but not a boundary in $\mathcal{C}_\infty^{\otimes 2}$. Hence our proof of Lemma 3.6 would fail.

3.3 Operad describing diagrams

Let a small category \mathbf{C} (together with its operadic version (4)) and a dg operad \mathcal{A} be given. A (\mathbf{C} -shaped) **diagram** of (\mathcal{A} -algebras) is a functor

$$D : \mathbf{C} \rightarrow \mathcal{A}\text{-algebras}.$$

Now we describe a dg V -operad \mathcal{D} such that \mathcal{D} -algebras are precisely \mathbf{C} -shaped diagrams. We denote by $*$ the free product of dg V -operad, i.e. the coproduct in the category of dg V -operad.

3.11 Definition. For any (noncoloured) dg operad $(\mathcal{A}, \partial_{\mathcal{A}})$, define

$$(\mathcal{A}, \partial_{\mathcal{A}})_{\mathbf{C}} := \left(\frac{(*_{v \in V} \mathcal{A}_v) * \mathcal{C}}{(fa_{I(f)} - a_{O(f)} f^{\otimes \text{ar}(a)} \mid a \in \mathcal{A}, f \in \text{Mor } \mathbf{C})}, \partial \right), \quad (11)$$

where \mathcal{A}_v is a copy of \mathcal{A} concentrated in colour v and symbols for its elements are decorated with lower index v . Let the differential ∂ be defined by formulas

$$\begin{aligned}\partial a_v &= (\partial_{\mathcal{A}} a)_v, \\ \partial f &= 0\end{aligned}$$

for any $a \in \mathcal{A}$, $v \in V$ and $f \in C$. For a dg operad morphism $(\mathcal{A}, \partial_{\mathcal{A}}) \xrightarrow{\xi} (\mathcal{B}, \partial_{\mathcal{B}})$, a dg V -operad morphism

$$(\mathcal{A}, \partial_{\mathcal{A}})_C \xrightarrow{\xi_C} (\mathcal{B}, \partial_{\mathcal{B}})_C$$

is defined by

$$\begin{aligned}\xi_C(a_v) &:= (\xi(a))_v, \\ \xi_C(f) &:= f.\end{aligned}\tag{12}$$

It is easy to verify that the defining ideal of \mathcal{A}_C is sent to the defining ideal of \mathcal{B}_C and also that $\xi_C \partial = \partial \xi_C$, thus ξ_C is well defined. It is also easily seen that

$$\xi_C \zeta_C = (\xi \zeta)_C$$

for any two dg operad morphisms ξ, ζ , hence

$$-_C : \text{dg operads} \rightarrow \text{dg } V\text{-operads}$$

is a functor.

Set

$$\mathcal{D} := (\mathcal{A}, \partial_{\mathcal{A}})_C.$$

It is immediately seen that the functor D above is essentially the same thing as \mathcal{D} -algebra, i.e. dg V -operad morphism $\mathcal{D} \rightarrow \mathcal{E}nd_W$, where $W = \bigoplus_{v \in V} D(v)$ and each $D(v)$ is a dg k -module of colour v .

The following lemma generalizes Proposition 5 of [7].

3.12 Lemma. $-_C$ preserves quisms.

Proof. Let $\mathcal{A} = (\mathcal{A}, \partial)$ be a dg V -operad and let $v, v_1, \dots, v_n \in V$. We claim that there is an isomorphism

$$\begin{aligned}\mathcal{A}_C \left(\begin{matrix} v \\ v_1, \dots, v_n \end{matrix} \right) &\cong \mathcal{A}_v(n) \otimes_{\Sigma_n} \left(\bigoplus_{\vec{W}} \mathcal{C} \left(\begin{matrix} v \\ w_{1,1} \end{matrix} \right) \otimes \dots \otimes \mathcal{C} \left(\begin{matrix} v \\ w_{n,1} \end{matrix} \right) \otimes k \langle \Sigma(\vec{W}, \vec{v}) \rangle \right) = \\ &= (\mathcal{A}_v \circ \mathcal{C}) \left(\begin{matrix} v \\ v_1, \dots, v_n \end{matrix} \right)\end{aligned}$$

of dg $k \langle \Sigma_{\vec{v}} \rangle$ -modules, where $\vec{W} = (w_{1,1}, \dots, w_{n,1}) \in V^n$.

The isomorphism assigns a *canonical form* to an element $x \in \mathcal{A}_C \left(\begin{matrix} v \\ v_1, \dots, v_n \end{matrix} \right)$: Assume x is an equivalence class of a composition of the generators from $(\ast_{v \in V} \mathcal{A}_v) \ast \mathcal{C}$. Now use the defining relations to “move” the generators from $\ast_{v \in V} \mathcal{A}_v$ to the left, so that $x = a \otimes f_1 \otimes \dots \otimes f_n \otimes \sigma$ for some $a \in \mathcal{A}$, $f_1, \dots, f_n \in \mathcal{C}$ and $\sigma \in \Sigma_n$. Then $a \otimes f_1 \otimes \dots \otimes f_n \otimes \sigma$ is called the canonical form of x . By freeness, it is uniquely determined

by x . It is immediate that we get an isomorphism of dg $k\langle\Sigma_{\vec{v}}\rangle$ -modules above and also $\mathcal{A}_{\mathcal{C}}(n) \cong (\mathcal{A}_v \circ \mathcal{C})(n)$ as dg $k\langle\Sigma_n\rangle$ -modules.

Let $(\mathcal{A}, \partial_{\mathcal{A}}) \xrightarrow{\xi} (\mathcal{B}, \partial_{\mathcal{B}})$ be a quism. It is easy to see that the following diagram commutes

$$\begin{array}{ccc} \mathcal{A}_{\mathcal{C}}\left(\begin{array}{c} v \\ v_1, \dots, v_n \end{array}\right) & \xrightarrow{\xi_{\mathcal{C}}} & \mathcal{B}_{\mathcal{C}}\left(\begin{array}{c} v \\ v_1, \dots, v_n \end{array}\right) \\ \cong \downarrow & & \downarrow \cong \\ (\mathcal{A}_v \circ \mathcal{C})\left(\begin{array}{c} v \\ v_1, \dots, v_n \end{array}\right) & \xrightarrow{\xi \otimes 1 \otimes \dots \otimes 1 \otimes 1} & (\mathcal{B}_v \circ \mathcal{C})\left(\begin{array}{c} v \\ v_1, \dots, v_n \end{array}\right) \end{array}$$

The diagram descends to homology, the lower horizontal arrow becomes an isomorphism by Lemma 2.8, thus the upper horizontal arrow becomes an isomorphism as well. \square

3.4 Main theorem

Suppose we are given a resolution

$$\mathcal{C}_{\infty} = (\mathbb{F}(F), \partial) \xrightarrow[\phi_{\mathcal{C}}]{\sim} (\mathcal{C}, 0)$$

of \mathcal{C} satisfying the *assumptions* 3.1 and a *minimal* resolution

$$\mathcal{A}_{\infty} = (\mathbb{F}(X), \partial) \xrightarrow[\phi_{\mathcal{A}}]{\sim} (\mathcal{A}, \partial)$$

of \mathcal{A} . We will use the same symbol ∂ for all the involved differentials. The correct meaning will always be clear from the context. Denote

$$\begin{aligned} X_V &:= X \otimes k\langle V \rangle \\ X_F &:= \uparrow X \otimes F. \end{aligned}$$

These are V - Σ -modules by Σ action on the X factor. An element $x \otimes v \in X \otimes k\langle V \rangle$ is denoted by x_v . Analogously, $\uparrow x \otimes f \in \uparrow X \otimes F$ is denoted x_f . Hence

$$|x_v| = |x|, \quad |x_f| = |x| + |f| + 1.$$

Obviously $X_V = \bigoplus_{v \in V} X \otimes k\langle v \rangle$ and for any $v \in V$ we denote

$$X_v := X \otimes k\langle v \rangle.$$

Finally, let

$$\mathcal{D}_{\infty} := \mathbb{F}(X_V \oplus F \oplus X_F).$$

We also extend the notation x_v for $x \in X$ and $v \in k\langle V \rangle$ to an operad morphism

$$\begin{aligned} -_v : \mathbb{F}(X) &\rightarrow \mathbb{F}(X_v) \hookrightarrow \mathcal{D}_{\infty} \\ x &\mapsto x_v. \end{aligned}$$

We will be interested in differentials of a special form on \mathcal{D}_{∞} . To state it precisely, we introduce the following maps:

3.13 Definition. For any $x \in X(n)$, the linear map

$$\mathcal{P}(x, -) : \mathcal{C}_\infty \rightarrow \mathcal{D}_\infty(n)$$

is uniquely given by requiring

$$\begin{aligned} \mathcal{P}(x, f) &= x_f, \\ \mathcal{P}(x, r_1 r_2) &= \mathcal{P}(x, r_1) \llbracket r_2 \rrbracket_n + (-1)^{|r_1|(|x|+1)} r_1 \mathcal{P}(x, r_2) \end{aligned}$$

for every $f \in F$ and $r_1, r_2 \in \mathcal{C}_\infty$.

Thus $\mathcal{P}(x, -)$ behaves much like a derivation of degree $|x| + 1$. Checking it is well defined boils down to verify $\mathcal{P}(x, r_1(r_2 r_3)) = \mathcal{P}(x, (r_1 r_2) r_3)$, which is easy. Note that $\mathcal{P}(x, 1) = 0$ for any unit in the V -operad \mathcal{C}_∞ .

3.14 Definition. Let \mathcal{A} be a graded operad. Recall that a presentation

$$\frac{\mathbb{F}(E)}{(R)} \cong \mathcal{A} \tag{13}$$

is called *quadratic* iff $R \subset \mathbb{F}^2(E)$, i.e. elements of R are sums of operadic compositions of exactly 2 generators from E . The elements of the Σ -module E are called *generating operations*.

Recall \mathcal{A} is called *Koszul* iff there is a quadratic presentation (13) such that the cobar construction on the Koszul dual \mathcal{A}^i of \mathcal{A} is a resolution of \mathcal{A} , i.e.

$$\Omega(\mathcal{A}^i) \xrightarrow[\phi_{\mathcal{A}}]{\sim} (\mathcal{A}, 0).$$

See [5] for the notation and more details.

We are now finally able to state our main result:

3.15 Theorem. Let \mathcal{A} be a *Koszul* operad with generating operations concentrated in a *single arity* ≥ 2 and a *single degree* ≥ 0 . Let \mathbf{C} be a small category and let $(\mathcal{C}_\infty, \partial_{\mathcal{C}}) \xrightarrow{\phi_{\mathcal{C}}} (\mathcal{C}, 0)$ be its resolution (in the sense explained in Section 3.1) satisfying the assumptions 3.1. Then the graded V -operad $\mathcal{D} = (\mathcal{A}, 0)_{\mathbf{C}}$ of (11), describing \mathbf{C} -shaped diagrams of \mathcal{A} -algebras, has a free resolution

$$(\mathcal{D}_\infty, \partial) \xrightarrow[\Phi]{\sim} (\mathcal{D}, 0)$$

of the form

$$\mathcal{D}_\infty := \mathbb{F}(X_V \oplus F \oplus X_F)$$

with the differential ∂ given by

$$\begin{aligned} \partial x_v &= (\partial x)_v, \\ \partial f &= \partial_{\mathcal{C}} f, \\ \partial x_f &= (-1)^{1+|x|} \mathcal{P}(x, \partial f) + (-1)^{1+|x||f|} f x_{I(f)} + x_{O(f)} \llbracket f \rrbracket_n + \omega(x, f), \end{aligned} \tag{14}$$

where $x \in X(n)$, $v \in V$, $f \in F$ and $\omega(x, f)$ lies in the arity n part of the

$$\text{ideal } \mathcal{I}^{<n} \text{ generated by } F_{\geq 1} \oplus X_F(<n) \tag{15}$$

in

$$\mathcal{D}_\infty^{<n} := \mathbb{F}(X_V(<n) \oplus F \oplus X_F(<n)). \quad (16)$$

The differential ∂ on \mathcal{D}_∞ is minimal iff ∂ on \mathcal{C}_∞ is. The dg V -operad morphism Φ is given by

$$\begin{aligned} \Phi(x_v) &= (\phi_{\mathcal{A}}(x))_v, \\ \Phi(f) &= \phi_{\mathcal{C}}(f), \\ \Phi(x_f) &= 0. \end{aligned}$$

3.16 Remark. This is a weaker form of Conjecture 31 of [7]. First, we are restricted to Koszul operad \mathcal{A} with generating operations in a single arity and a single degree, while the conjecture lets \mathcal{A} be any dg operad. Second, the ideal $\mathcal{I}^{<n}$ is larger, generated by $F_{\geq 1} \oplus X_F(<n)$, while the conjectured

$$\text{ideal } \mathcal{I}_{\text{orig}}^{<n} \text{ is generated just by } X_F(<n). \quad (17)$$

In particular, we recover, at least for \mathcal{A} as above, Theorem 7 of [7] dealing with the case of \mathcal{C} being a single morphism between two distinct objects and \mathcal{C}_∞ its trivial resolution. Observe that in this case, $\mathcal{I}^{<n}$ is in fact generated just by $X_F(<n)$ since $F_{\geq 1} = 0$ (of course, similar statement holds for any \mathcal{C}_∞ concentrated in degree 0, which corresponds to a *free* category \mathcal{C}). We also recover Theorems 18 and 24 of [7], again with the above mentioned restrictions.

However, there seems to be a completely unclear statement at the very end of the proof of Theorem 7, page 11 of [7]. As the proofs of Theorems 18 and 24 of [7] are only sketched, there is probably the same problem. To remedy it, we had to introduce our assumptions. We will discuss these assumptions in detail after proving our main theorem. However, we don't know any counterexample to the original theorems of [7].

4 Proof of main theorem

4.1 Lemmas

4.1 Lemma. Let \mathcal{C} be a small category, let $(\mathcal{C}_\infty, \partial_{\mathcal{C}}) \xrightarrow{\phi_{\mathcal{C}}} (\mathcal{C}, 0)$ be its resolution satisfying the assumptions 3.1. For *any minimal* dg V -operad of the form $(\mathbb{F}(X), \partial)$ with $X(0) = X(1) = 0$, let

$$\mathcal{D}_\infty := \mathbb{F}(X_V \oplus F \oplus X_F)$$

and *assume* there is a differential ∂ on \mathcal{D}_∞ satisfying

$$\begin{aligned} \partial x_v &= (\partial x)_v, \\ \partial f &= \partial_{\mathcal{C}} f, \\ \partial x_f &= (-1)^{1+|x|} \mathcal{P}(x, \partial f) + (-1)^{1+|x||f|} f x_{I(f)} + x_{O(f)} \llbracket f \rrbracket_n + \omega(x, f), \end{aligned}$$

where $x \in X(n)$, $v \in V$, $f \in F$ and

$$\omega(x, f) \in \mathcal{D}_\infty^{<n} = \mathbb{F}(X_V(<n) \oplus F \oplus X_F(<n))(n).$$

Assume ϕ is a dg V -operad morphism

$$(\mathcal{D}_\infty, \partial) \xrightarrow{\phi} (\mathbb{F}(X), \partial)_\mathbb{C}$$

satisfying

$$\begin{aligned}\phi(x_v) &= x_v, \\ \phi(f) &= f\end{aligned}\tag{18}$$

for $f \in F_0$ and vanishing on all the other generators. Then ϕ is a quism.

We use the symbol x_v either for $x_v \in X_V \subset \mathcal{D}_\infty$ or $x_v \in \mathbb{F}(X)_v \subset (\mathbb{F}(X), \partial)_\mathbb{C}$. Similarly for $f \in F_0$. The correct meaning will always be clear from the context.

Proof. Let \mathfrak{F}_i be the sub V - Σ -module of \mathcal{D}_∞ spanned by free compositions containing at least $-i$ generators from $X_V \oplus X_F$. \mathfrak{F}_i 's form a filtration

$$\cdots \subset \mathfrak{F}_{-2} \subset \mathfrak{F}_{-1} \subset \mathfrak{F}_0 = \mathcal{D}_\infty.$$

$\mathcal{P}(x, \partial f) \in \mathfrak{F}_{-1}$ is obvious and $\text{ar}(\omega(x, f)) = \text{ar}(x) \geq 2$ implies $\omega(x, f) \in \mathfrak{F}_{-1}$. Hence $\partial \mathfrak{F}_i \subset \mathfrak{F}_i$. Since X contains no elements of arity 0 and 1, for a fixed arity n the arity n part $\mathfrak{F}_i(n)$ of this filtration is bounded below. Consider the corresponding spectral sequence $(E^*(n), \partial^*(n))$. For each n , $(E^*(n), \partial^*(n))$ converges by the classical convergence theorem. We collect these spectral sequences into (E^*, ∂^*) . Recall that each (E^i, ∂^i) is a dg V -operad. In the sequel, such arity-wise constructions will be understood without mentioning the arity explicitly. For the 0th term, we have

$$E^0 \cong \mathcal{D}_\infty$$

as graded V -operad. Now we make ∂^0 explicit. Let $x \in X(n)$, $n \geq 2$. By the minimality, each summand of ∂x_v contains at least 2 generators from X_V , hence $\partial x_v \in \mathfrak{F}_{-2}$ and $\partial^0 x_v = 0$. Next, observe that for $n = 2$, $\omega(x, f) = 0$ by arity reasons. Let $n \geq 3$. Each summand of $\omega(x, f) \neq 0$ contains only generators of arity $< n$, hence at least 2 of these are of arities ≥ 2 . But generators of arity ≥ 2 come from $X_V \oplus X_F$, i.e. $\omega(x, f) \in \mathfrak{F}_{-2}$. Hence the differential ∂^0 is the derivation determined by formulas

$$\begin{aligned}\partial^0 x_v &= 0 \\ \partial^0 f &= \partial f \\ \partial^0 x_f &= (-1)^{1+|x|} \mathcal{P}(x, \partial f) + (-1)^{1+|x||f|} f x_{I(f)} + x_{O(f)} \llbracket f \rrbracket_n\end{aligned}$$

for $x \in X(n)$ and $f \in F$.

There is a similar construction on $(\mathbb{F}(X), \partial)_\mathbb{C}$. Denote ∂' its differential. Let \mathfrak{F}'_i be the sub V - Σ -module of $(\mathbb{F}(X), \partial)_\mathbb{C}$ spanned by free compositions containing at least $-i$ generators from X_V . Then these form a filtration

$$\cdots \subset \mathfrak{F}'_{-2} \subset \mathfrak{F}'_{-1} \subset \mathfrak{F}'_0 = (\mathbb{F}(X), \partial)_\mathbb{C}.$$

Obviously $\partial' \mathfrak{F}'_i \subset \mathfrak{F}'_i$. By the same argument as above, this filtration is bounded below and hence the corresponding spectral sequence (E'^*, ∂'^*) converges. For the 0th page, we have

$$(E'^0, \partial'^0) \cong (\mathbb{F}(X), 0)_\mathbb{C}$$

as dg V -operad, i.e.

$$\partial'^0 = 0.$$

The dg V -operad morphism ϕ satisfies $\partial\mathfrak{F}_i \subset \mathfrak{F}'_i$, hence it induces a morphism $\phi^* : (E^*, \partial^*) \rightarrow (E'^*, \partial'^*)$ of spectral sequences. By [12], Theorems 5.2.12 and 5.5.1, to prove that ϕ is quism, it suffices to show that ϕ^0 is a quism. We will prove

$$H_*(E^0, \partial^0) = \frac{\mathbb{F}(X_V \oplus F_0)}{(\{-fx_{I(f)} + x_{O(f)}\llbracket f \rrbracket_{\text{ar}(x)} \mid x \in X, f \in F_0\} \cup \partial F_1)}, \quad (19)$$

compare with (5) and Definition 3.11. This implies $H_*(\phi^0)$ is the identity and we are done.

The dg V -operad (E^0, ∂^0) carries a filtration

$$0 = \mathfrak{F}''_{-1} \subset \mathfrak{F}''_0 \subset \mathfrak{F}''_1 \subset \cdots,$$

where \mathfrak{F}''_i is sub V - Σ -module of E^0 spanned by compositions with

$$(\text{degree} + \text{number of generators from } X_V) \leq i.$$

Obviously $\partial^0 \mathfrak{F}''_i \subset \mathfrak{F}''_i$. This filtration is bounded below and exhaustive, hence the corresponding spectral sequence (E^{0*}, ∂^{0*}) converges by [12], Theorem 5.2.12. We have

$$E^{00} \cong \mathcal{D}_\infty$$

as graded V -operad and

$$\begin{aligned} \partial^{00} x_v &= 0, \\ \partial^{00} f &= 0, \\ \partial^{00} x_f &= (-1)^{1+|x||f|} f x_{I(f)} + x_{O(f)} \llbracket f \rrbracket_n \end{aligned}$$

for $x \in X(n)$ and $f \in F$. We will show

$$H_*(E^{00}, \partial^{00}) = \frac{\mathbb{F}(X_V \oplus F)}{((-1)^{1+|x||f|} f x_{I(f)} + x_{O(f)} \llbracket f \rrbracket_{\text{ar}(x)} \mid x \in X, f \in F)}. \quad (20)$$

Assume this is already done and let's prove (19). We proceed to the 1st page E^{01} of E^{0*} :

$$E^{01} \cong H_*(E^{00}, \partial^{00})$$

and under this isomorphism, ∂^{01} is given by

$$\begin{aligned} \partial^{01} x_v &= 0, \\ \partial^{01} f &= \partial f. \end{aligned}$$

By the same argument as in the proof of Lemma 3.12,

$$(E^{01}, \partial^{01}) \cong (\mathbb{F}(X_V), 0) \circ (\mathbb{F}(F), \partial).$$

By Lemma 2.8 and (5),

$$H_*(E^{01}, \partial^{01}) \cong \mathbb{F}(X_V) \circ \mathcal{C} \cong \mathbb{F}(X_V) \circ \frac{\mathbb{F}(F_0)}{(\partial F_1)}$$

and by the argument of Lemma 3.12 again,

$$H_*(E^{01}, \partial^{01}) \cong \frac{\mathbb{F}(X_V \oplus F_0)}{(\{-fx_{I(f)} + x_{O(f)}\llbracket f \rrbracket_{\text{ar}(x)} \mid x \in X, f \in F_0\} \cup \partial F_1)}. \quad (21)$$

This is the 2nd page E^{02} and we claim that all the higher differentials vanish: $\partial^{0k} = 0$ for $k \geq 2$. To see this, let's assign *inner degree*, denoted by $\| - \|$, to generators of E^0 :

$$\|x_v\| = 0, \quad \|f\| = |f|, \quad \|x_f\| = |f| + 1.$$

This extends to E^0 by requiring the operadic composition to be of inner degree 0. Now notice that ∂^0 is of inner degree -1 and so are all the differentials ∂^{0k} . But (21) is concentrated in inner degree 0, hence the spectral sequence (E^{0*}, ∂^{0*}) collapses as claimed. We conclude that $E^{02} = E^{0\infty} \cong H_*(E^0, \partial^0)$, thus proving (19).

It remains to prove (20). Let \mathfrak{F}_i''' be the sub V - Σ -module of $E^{00} \cong \mathcal{D}_\infty$ spanned by compositions with at least $-i$ generators from $F \oplus X_F$. Then

$$\cdots \subset \mathfrak{F}_{-2}''' \subset \mathfrak{F}_{-1}''' \subset \mathfrak{F}_0''' = E^{00}$$

is a filtration with $\partial^{00}\mathfrak{F}_i''' \subset \mathfrak{F}_i'''$. Denote $(E^{00*}, \partial^{00*})$ the corresponding spectral sequence. The convergence of this spectral sequence will be discussed later. We have

$$\begin{aligned} E^{000} &\cong \mathcal{D}_\infty, \\ \partial^{000}x_v &= 0, \\ \partial^{000}f &= 0, \\ \partial^{000}x_f &= (-1)^{1+|x||f|}fx_{I(f)}. \end{aligned}$$

Now we prove

$$H_*(E^{000}, \partial^{000}) = \frac{\mathbb{F}(X_V \oplus F)}{((-1)^{1+|x||f|}fx_{I(f)} \mid x \in X, f \in F)}. \quad (22)$$

First observe that $\mathbb{F}(F) \circ (X_V \oplus X_F)$ is closed under ∂^{000} and

$$H_*(\mathbb{F}(F) \circ (X_V \oplus X_F), \partial^{000}) = X_V. \quad (23)$$

In the sequel, we may drop the differentials from the notation “ $H_*(-, \partial)$ ” if no confusion can arise. Let

$$\begin{aligned} P_0 &:= k\langle 1 \rangle, \\ P_{n+1} &:= \mathbb{F}(F) \oplus \mathbb{F}(F) \circ (X_V \oplus X_F) \circ P_n \end{aligned}$$

for $n \geq 0$. We immediately see that P_n 's are closed under ∂^{000} and

$$P_n = \bigoplus_{i=0}^{n-1} (\mathbb{F}(F) \circ (X_V \oplus X_F))^{oi} \circ \mathbb{F}(F) \oplus (\mathbb{F}(F) \circ (X_V \oplus X_F))^{on}, \quad (24)$$

where we used the (iterated) composition product (2). By Lemma 2.8 and (23),

$$H_*(P_n) \cong \bigoplus_{i=0}^{n-1} (X_V)^{oi} \circ \mathbb{F}(F) \oplus (X_V)^{on}.$$

(24) provides a chain of inclusions

$$P_0 \hookrightarrow P_1 \hookrightarrow \cdots \rightarrow \varinjlim_n P_n \cong E^{000}$$

with direct limit E^{000} , as easily seen. Since direct limits commute with homology,

$$\begin{aligned} H_*(E^{000}, \partial) &\cong \varinjlim_i H_*(P_n) = \varinjlim_i \left(\bigoplus_{i=0}^{n-1} (X_V)^{\circ i} \circ \mathbb{F}(F) \oplus (X_V)^{\circ n} \right) \cong \\ &\cong \mathbb{F}(X_V) \circ \mathbb{F}(F) \cong \frac{\mathbb{F}(X_V \oplus F)}{((-1)^{1+|x||f|} f x_{I(f)} \mid x \in X, f \in F)}. \end{aligned}$$

The 1st page E^{001} is therefore described by (22). An argument with inner degree analogous to the one above shows that $\partial^{00k} = 0$ for $k \geq 1$: In (E^{00}, ∂^{00}) , set

$$\|x_v\| = \|f\| = 0, \quad \|x_f\| = 1.$$

Hence $E^{001} = E^{00\infty}$ is the stable term.

Although we don't know how to prove the convergence of the spectral sequence $(E^{00*}, \partial^{00*})$ directly (the filtration is bounded above but not below, we only have the Hausdorff property $\cap_i \mathfrak{F}_i''' = 0$), there is a weaker statement which follows from Lemma 5.5.7 of [12]: The i^{th} graded part $\mathfrak{F}_i''' H_*(E^{00}, \partial^{00}) / \mathfrak{F}_{i-1}''' H_*(E^{00}, \partial^{00})$ of the filtration on homology⁶ is isomorphic to a subspace e_i of $E_i^{00\infty}$.⁷ We have (simplifying the notation)

$$\begin{aligned} \frac{\mathbb{F}(X_V \oplus F)}{((-1)^{1+|x||f|} f x_{I(f)} + x_{O(f)} \llbracket f \rrbracket_{\text{ar}(x)})} &\subset H_*(E^{00}) \cong \bigoplus_{i \leq 0} \frac{\mathfrak{F}_i''' H_*(E^{00})}{\mathfrak{F}_{i-1}''' H_*(E^{00})} \cong \bigoplus_{i \leq 0} e_i \subset \bigoplus_{i \leq 0} E_p^\infty \cong \\ &\cong \frac{\mathbb{F}(X_V \oplus F)}{((-1)^{1+|x||f|} f x_{I(f)})}, \end{aligned}$$

where the first inclusion is the obvious part of (20) and the second inclusion has just been discussed. It is not difficult to map the left-hand side through all the isomorphisms and to see that it is mapped *onto* the right-hand side. Hence the first inclusion is in fact equality and we are done proving (20) and consequently the whole Lemma 4.1. \square

4.2 Lemma. Let $\omega(x, f)$ of Lemma 4.1 moreover satisfies $\omega(x, f) \in \mathcal{I}^{<n}$ (recall (15)). Then ϕ , uniquely determined by (18) as a graded V -operad morphism, is automatically a dg V -operad morphism (i.e. ϕ commutes with the differentials).

Proof. We have to verify $\phi\partial = \partial\phi$ for generators from $X_V \oplus F \oplus X_F$. The only nontrivial case concerns X_F : we have to verify $\phi\partial x_f = 0$. We have

$$\begin{aligned} \phi\partial x_f &= (-1)^{1+|x|} \phi(\mathcal{P}(x, \partial f)) + \\ &+ (-1)^{1+|x||f|} \phi(f) \phi(x_{I(f)}) + \phi(x_{O(f)}) \phi(\llbracket f \rrbracket) + \phi(\omega(x, f)). \end{aligned}$$

If $\partial f = 0$, then the first term vanishes trivially. If $\partial f \neq 0$, then each summand of $\mathcal{P}(x, \partial f)$ contains a generator from X_F and hence the first term vanishes too.

By the definition of $\mathcal{I}^{<n}$, we have $\phi(\omega(x, f)) = 0$.

⁶Recall the usual notation $\mathfrak{F}_i''' H_*(E^{00}, \partial^{00}) := \text{Im}(H_*(\mathfrak{F}_i''', \partial^{00}) \rightarrow H_*(E^{00}, \partial^{00}))$.

⁷The lower index denotes the grading associated to the filtration \mathfrak{F}_i''' .

Hence it remains to prove

$$(-1)^{1+|x||f|}\phi(f)\phi(x_{I(f)}) + \phi(x_{O(f)})\phi(\llbracket f \rrbracket) = 0.$$

If $|f| > 0$, we have $\phi f = 0$ by definition. Also $|\llbracket f \rrbracket| > 0$ and hence each summand of $\llbracket f \rrbracket$ contains a generator from $F_{\geq 1}$ and consequently $\phi(\llbracket f \rrbracket) = 0$. For $|f| = 0$, we may assume $f \in M$ and we want to prove $-fx_{I(f)} + x_{O(f)}f^{\otimes n} = 0$ in $(\mathbb{F}(X), \partial)_{\mathcal{C}}$. But this is exactly one of the defining relations of $(\mathbb{F}(X), \partial)_{\mathcal{C}}$. \square

4.3 Lemma. Let an operad \mathcal{A} be Koszul with generating operations concentrated in a single arity $N \geq 2$ and a single degree $D \geq 0$. Then for every generator x of the minimal resolution of \mathcal{A} there is $k \geq 1$ such that

$$\text{ar}(x) = a_k := 1 + (N - 1)k, \quad |x| = d_k := -1 + (D + 1)k.$$

Moreover, there is K (possibly $K = +\infty$) such that a generator of arity a_k and degree d_k exists iff $k < K$.

Proof. By Koszulity, we have the minimal resolution

$$\Omega(\mathcal{A}^i) \xrightarrow{\sim} (\mathcal{A}, 0)$$

given by the cobar construction $\Omega(\mathcal{A}^i) = (\mathbb{F}(\downarrow \overline{\mathcal{A}^i}), \partial)$. Assume \mathcal{A} has the quadratic presentation (13). Recall that the Koszul dual \mathcal{A}^i is the quadratic cooperad cogenerated by $\uparrow E$ with corelations $\uparrow^2 E$, see [5], 7.1.4. Thus \mathcal{A}^i is a sub Σ -module of $\mathbb{F}(\uparrow E)$, hence it is concentrated in arities $1 + (N - 1)k$ and degrees $(D + 1)k$. Hence $\downarrow \overline{\mathcal{A}^i}$ is concentrated in arities $a_k = 1 + (N - 1)k$ and degrees $d_k = -1 + (D + 1)k$.

We give only a brief proof of the last claim of this lemma, since we won't need it in the sequel. Suppose that for every $k < K$ a generator of arity a_k and degree d_k exists. Further let there be no generator in arity a_K . By the inductive construction of the minimal resolution, as described in the proof of Theorem 3.125 of [10], the generators in the next possible arity a_{K+1} have degree $\leq d_{K-1} + 2D + 1 = d_{K+1} - 1$. But the existence of any such generator would contradict the previous part of this lemma. In the next arity, a_{K+2} , the generators would have to have degree $\leq d_{K-1} + 3D + 1 < d_{K+2}$. And so on, hence there are no generators in arity a_k for $k \geq K$. We encourage the reader to go through the cases $D = 0$ and $D = 1$. \square

4.4 Lemma. Let an operad \mathcal{A} be Koszul with generating operations concentrated in a single arity ≥ 2 and a single degree ≥ 0 . Then for every $x \in X$ and $f \in F$, there is $\omega(x, f) \in \mathcal{I}^{<\text{ar}(x)}$ as stated in Theorem 3.15, i.e. the derivation ∂ defined by (14) is indeed a differential on \mathcal{D}_{∞} .

To prove this lemma, it is convenient to extend $\omega(x, f)$'s to a linear map as follows. Fix $x \in X(n)$. The linear map

$$\omega(x, -) : \mathcal{C}_{\infty} \rightarrow \mathcal{D}_{\infty}(n)$$

is uniquely determined by

$$\begin{aligned} & \text{(arbitrary) values } \omega(x, f) \text{ on } f \in F \text{ and} \\ & \omega(x, r_1 r_2) = \omega(x, r_1) \llbracket r_2 \rrbracket_n + (-1)^{|r_1||x|} r_1 \omega(x, r_2) \end{aligned}$$

for any $r_1, r_2 \in \mathcal{C}_{\infty}$.

Thus $\omega(x, -)$ behaves much like a derivation of degree $|x|$. Checking it is well defined is similar to 3.13.

4.5 Lemma. For any $r \in \mathcal{C}_\infty$, the formula (14) with r in place of f still holds:

$$\partial \mathcal{P}(x, r) = (-1)^{1+|x|} \mathcal{P}(x, \partial r) + (-1)^{1+|r||x|} r x_{I(r)} + x_{O(r)} \llbracket r \rrbracket_n + \omega(x, r).$$

The proof explains the \pm signs in the definition (14) of ∂ on \mathcal{D}_∞ .

Proof. It suffices to prove the lemma for r of the form $r = f_1 f_2 \cdots f_k$, where $f_i \in F$. We proceed by induction on k . The case $k = 1$ is exactly formula (14). Let $k \geq 2$ and suppose the lemma holds for every sum of compositions of at most $k - 1$ elements and let $r = r_1 r_2$, where r_1, r_2 are compositions of at most $k - 1$ generators from F . Now we want to prove

$$\begin{aligned} \partial \mathcal{P}(x, r_1 r_2) &= \\ &= (-1)^{1+|x|} \mathcal{P}(x, \partial(r_1 r_2)) + (-1)^{1+(|r_1|+|r_2|)|x|} r_1 r_2 x_{I(r_2)} + x_{O(r_1)} \llbracket r_1 r_2 \rrbracket_n + \omega(x, r_1 r_2). \end{aligned}$$

It is a straightforward computation, we will compare Left-Hand Side and Right-Hand Side:

$$\begin{aligned} \text{LHS} &= \partial \left(\mathcal{P}(x, r_1) \llbracket r_2 \rrbracket_n + (-1)^{|r_1|(|x|+1)} r_1 \mathcal{P}(x, r_2) \right) = \\ &= \left((-1)^{1+|x|} \mathcal{P}(x, \partial r_1) + (-1)^{1+|r_1||x|} r_1 x_{I(r_1)} + x_{O(r_1)} \llbracket r_1 \rrbracket_n + \omega(x, r_1) \right) \llbracket r_2 \rrbracket_n + \\ &\quad + (-1)^{|x|+|r_1|+1} \mathcal{P}(x, r_1) \llbracket \partial r_2 \rrbracket_n + (-1)^{|r_1|(|x|+1)} (\partial r_1) \mathcal{P}(x, r_2) + \\ &\quad + (-1)^{|r_1|(|x|+1)+|r_1|} r_1 \left((-1)^{1+|x|} \mathcal{P}(x, \partial r_2) + (-1)^{1+|r_2||x|} r_2 x_{I(r_2)} + \right. \\ &\quad \left. + x_{O(r_2)} \llbracket r_2 \rrbracket_n + \omega(x, r_2) \right) \\ \text{RHS} &= (-1)^{1+|x|} \mathcal{P}(x, (\partial r_1) r_2) + (-1)^{1+|x|+|r_1|} \mathcal{P}(x, r_1 \partial r_2) + \\ &\quad + (-1)^{1+(|r_1|+|r_2|)|x|} r_1 r_2 x_{I(r_2)} + x_{O(r_1)} \llbracket r_1 \rrbracket_n \llbracket r_2 \rrbracket_n + \\ &\quad + \omega(x, r_1) \llbracket r_2 \rrbracket_n + (-1)^{|x||r_1|} r_1 \omega(x, r_2) = \\ &= (-1)^{1+|x|} \mathcal{P}(x, \partial r_1) \llbracket r_2 \rrbracket_n + (-1)^{1+|x|+(|x|+1)(|r_1|-1)} (\partial r_1) \mathcal{P}(x, r_2) + \\ &\quad + (-1)^{1+|x|+|r_1|} \mathcal{P}(x, r_1) \llbracket \partial r_2 \rrbracket_n + (-1)^{1+|x|+|r_1|+(|x|+1)|r_1|} r_1 \mathcal{P}(x, \partial r_2) + \\ &\quad + (-1)^{1+(|r_1|+|r_2|)|x|} r_1 r_2 x_{I(r_2)} + x_{O(r_1)} \llbracket r_1 \rrbracket_n \llbracket r_2 \rrbracket_n + \\ &\quad + \omega(x, r_1) \llbracket r_2 \rrbracket_n + (-1)^{|x||r_1|} r_1 \omega(x, r_2) \end{aligned}$$

The proof is finished by a careful sign inspection. \square

Proof of Lemma 4.4. Let $x \in X(n)$ and $f \in F_d$. First, we make a preliminary computation using the formula of Lemma 4.5:

$$\begin{aligned} \partial^2 x_f &= (-1)^{1+|x|} \partial \mathcal{P}(x, \partial f) + (-1)^{1+|x||f|} (\partial f) x_{I(f)} + (-1)^{1+|x||f|+|f|} f \partial x_{I(f)} + \\ &\quad + (\partial x_{O(f)}) \llbracket f \rrbracket_n + (-1)^{|x|} x_{O(f)} \llbracket \partial f \rrbracket_n + \partial \omega(x, f) = \cdots \\ &= (-1)^{1+|x|} \omega(x, \partial f) + (-1)^{1+|x||f|+|f|} f \partial x_{I(f)} + (\partial x_{O(f)}) \llbracket f \rrbracket_n + \partial \omega(x, f) \end{aligned}$$

The condition $\partial^2 x_f = 0$ is equivalent to

$$\partial \omega(x, f) = (-1)^{|x|} \omega(x, \partial f) + (-1)^{|f|(|x|+1)} f \partial x_{I(f)} - (\partial x_{O(f)}) \llbracket f \rrbracket_n =: \varphi(x, f).$$

To construct $\omega(x, f)$ so that $\partial^2 x_f = 0$, we will inductively solve the equation

$$\partial\omega(x, f) = \varphi(x, f) \quad (25)$$

for unknown $\omega(x, f)$. We proceed by induction on arity n of x and simultaneously by induction on degree d of f .

For $n = N$ (the arity of the generating operations of \mathcal{A}) and $d = 0$, we have $\partial f = 0 = \partial x$, hence (25) becomes $\partial\omega(x, f) = 0$, which has the trivial solution.

Fix n and d . Assume we have already constructed $\omega(x, f) \in \mathcal{I}^{<\text{ar}(x)}$ for every $x \in X(< n)$ and f of any degree and also for $x \in X(n)$ and $f \in F_{<d}$. Let $x \in X(n)$ and $f \in F_d$. Observe that $\varphi(x, f) \in \mathcal{D}_\infty^{<n}$ (recall (16)) by the induction assumption and minimality. When we restrict $\phi : \mathcal{D}_\infty \rightarrow (\mathbb{F}(X), \partial)_\mathbb{C}$ to $\mathcal{D}_\infty^{<n}$, we get the graded V -operad morphism

$$\mathcal{D}_\infty^{<n} \xrightarrow{\phi} (\mathbb{F}(X(< n)), \partial)_\mathbb{C}$$

denoted by the same symbol. By the induction assumption, $\partial^2 = 0$ on $\mathcal{D}_\infty^{<n}$. By Lemma 4.2, ϕ is dg V -operad morphism. By Lemma 4.1, ϕ is a quism. In a moment, we will show

$$\partial\varphi(x, f) = 0, \quad (26)$$

$$\phi\varphi(x, f) = 0. \quad (27)$$

This will imply the existence of $\omega(x, f) \in \mathcal{D}_\infty^{<n}$ such that $\partial\omega(x, f) = \varphi(x, f)$. In fact, $\omega(x, f) \in \mathcal{I}^{<n}$. To see this, assume a summand S of $\omega(x, f)$ is a composition of generators none of which comes from X_F . Hence S is an operadic composition of $x_1, \dots, x_a \in X_V(< n)$ and $f_1, \dots, f_b \in F$. By a degree count, we now show that at least one of f_j 's lies in $F_{\geq 1}$. By Lemma 4.3, let x_i have arity $1 + (N + 1)k_i$ and degree $-1 + (D + 1)k_i$. We have

$$|x| + |f| = |\omega(x, f)| = |S| = \sum_{i=1}^a |x_i| + \sum_{j=1}^b |f_j| = \sum_i (-1 + (D + 1)k_i) + \sum_j |f_j|$$

hence

$$\sum_j |f_j| = |x| + |f| + a - (D + 1) \sum_i k_i. \quad (28)$$

Now

$$\text{ar}(x) = \text{ar}(S) = 1 + \sum_i (\text{ar}(x_i) - 1) = 1 + (N - 1) \sum_i k_i$$

hence

$$|x| = -1 + (D + 1) \sum_i k_i.$$

Substituting this into (28), we get

$$\sum_j |f_j| = |f| + a - 1.$$

We have the trivial estimate $|f| \geq 0$. Since $\text{ar}(f_j) = 1$ for any j and $\text{ar}(x_i) < n = \text{ar}(S)$ for any i , we have $a \geq 2$. Hence

$$\sum_j |f_j| \geq 1$$

and therefore one of f_j 's lies in $F_{\geq 1}$.

It remains to verify the conditions (26), (27). For (26), we have

$$\partial\varphi(x, f) = (-1)^{|x|}\partial\omega(x, \partial f) + (-1)^{|f|(|x|+1)}(\partial f)\partial x_{I(f)} + (-1)^{|x|}(\partial x_{O(f)})\llbracket\partial f\rrbracket_n.$$

Lemma 4.5 and the induction hypothesis imply

$$\partial\omega(x, \partial f) = (-1)^{|x|(|f|-1)+|f|-1}(\partial f)\partial x_{I(f)} - (\partial x_{O(f)})\llbracket\partial f\rrbracket_n$$

and after substituting this into the previous equation, we get $\partial\varphi(x, f) = 0$.

For (27), let $d = 0$ first. Then $\varphi(x, f) = f\partial x_{I(f)} - (\partial x_{O(f)})\llbracket f\rrbracket_n$, hence we have to verify

$$f\partial x_{I(f)} - (\partial x_{O(f)})f^{\otimes n} = 0 \quad \text{in } (\mathbb{F}(X(< n)), \partial)_{\mathbb{C}}.$$

This follows by the same argument as Lemma 3.12. Now let $d > 0$. By induction assumption, $\omega(x, \partial f) \in \mathcal{I}^{<n}$ and therefore $\phi\omega(x, \partial f) = 0$. Finally $\phi f = 0 = \phi\llbracket f\rrbracket_n$ by definition of ϕ since $|f| = |\llbracket f\rrbracket_n| = d > 0$. \square

Now we can finally prove the main theorem:

of Theorem 3.15. Decompose Φ into

$$(\mathcal{D}_{\infty}, \partial) \xrightarrow{\phi} (\mathbb{F}(X), \partial)_{\mathbb{C}} \xrightarrow{(\phi_{\mathcal{A}})_{\mathbb{C}}} (\mathcal{A}, 0)_{\mathbb{C}} = (\mathcal{D}, 0).$$

The dg V -operad morphism $(\phi_{\mathcal{A}})_{\mathbb{C}}$ (recall Definition 3.11) is a quism by Lemma 3.12. ϕ is the graded V -operad morphism of Lemma 4.1. By Lemma 4.4, there are $\omega(x, f)$'s in $\mathcal{I}^{<\text{ar}(x)}$ such that ∂ on \mathcal{D}_{∞} is indeed a differential. By Lemma 4.2, ϕ is a dg V -operad morphism and finally, by Lemma 4.1, ϕ is a quism. \square

4.2 Discussion

It is a remarkable observation that in many cases, only a “principal” part of the differential determines what the homology is. This was exploited in [7] and also e.g. in [6] to partially resolve the PROP for bialgebras. Lemma 4.1 is an application of this principle. Here the minimality of \mathcal{A}_{∞} and the mild assumption $\omega(x, f) \in \mathcal{D}_{\infty}^{<n}$ (which in fact only formalizes what we mean by the principal part) are crucial for the spectral sequence argument to separate the principal part of ∂ . Apart from the minimality, arbitrary \mathcal{A}_{∞} with $X(0) = X(1) = 0$ is allowed (unfortunately, this excludes e.g. unital algebras). Notice, however, that we *assume* that ϕ commutes with differentials.

To guarantee this, we need a stronger constraint on $\omega(x, f)$. An easy sufficient way to ensure this is described in Lemma 4.2. It leads to the definition (15) of $\mathcal{I}^{<n}$.

Next, we have to construct a differential ∂ on \mathcal{D}_{∞} such that the assumptions of Lemma 4.2 are satisfied. This is achieved in Lemma 4.4. To begin with, one obtains $\omega(x, f) \in \mathcal{D}_{\infty}^{<n}$ by an inductive argument on the arity of the generators from X using Lemma 4.1. Then we have to improve this result. This is where the proof of Theorem 7 of [7] is unclear. We were not able to get the originally desired result $\omega(x, f) \in \mathcal{I}_{\text{orig}}^{<n}$ (recall (17)). But if one is able to control the interplay between arity and degree of the generators from X in a suitable way, one obtains at least $\omega(x, f) \in \mathcal{I}^{<n}$ by a simple degree count. A sufficient control is achieved for the Koszul resolution of a Koszul operad with generating operations bound in a single arity and degree. This

is explained in Lemma 4.3. We note that Lemma 4.4 can be proved under a weaker control over X , but the resulting conditions don't seem to be of any practical interest.

Still, it might be possible to improve the proof of Lemma 4.4 to get $\omega(x, f) \in \mathcal{I}_{\text{orig}}^{<n}$ even without the restrictions imposed on \mathcal{A} , thus proving the original Conjecture 31 of [7]. However, to our best knowledge, explicit examples of resolutions of diagrams $\mathcal{D} = \mathcal{A}_{\mathbb{C}}$ are known only for free categories \mathbb{C} and for operads satisfying the assumptions of Theorem 3.15. Moreover, in these cases $\mathcal{I}^{<n} = \mathcal{I}_{\text{orig}}^{<n}$. Hence these do not decide whether the conjecture is still plausible.

Notice a slightly stronger statement about what generators are needed to compose $\omega(x, f)$ can be made. For example, if $|f| = 0$, then $\omega(x, f)$ lies in the ideal generated by $X_f(<n)$ in $\mathbb{F}(X_{O(f)}(<n) \oplus X_{I(f)}(<n) \oplus k\langle f \rangle)$. This can be deduced from the proof of Lemma 4.4. However this doesn't seem to be important.

Finally notice that Lemma 4.1 is already quite a big achievement - it reduces the problem of resolving \mathcal{D} to finding $\omega(x, f)$'s from $\mathcal{D}_{\infty}^{<\text{ar}(x)}$ so that $\partial^2 = 0$ and the differential commutes with ϕ . Alternatively, by Lemma 4.2, the problem is reduced to finding $\omega(x, f)$'s from $\mathcal{I}^{<\text{ar}(x)}$ so that $\partial^2 = 0$.

5 Bar-cobar resolution of \mathcal{C}

Now we make the content of Theorem 3.15 more explicit in the case $\mathcal{C}_{\infty} = \Omega BC$. We apply Lemma 3.9 on the bar-cobar resolution $\mathcal{C}_{\infty} = \Omega BC$. Denote

$$\Sigma^n := \left\{ (\overleftarrow{f_n} \dots \overleftarrow{f_1}) \in (\text{Mor } \mathcal{C})^{\times n} \mid O(f_i) = I(f_{i+1}) \text{ for } 1 \leq i \leq n-1 \right\}$$

the set of chains of composable morphisms in \mathbb{C} of length n , e.g. $\Sigma^1 = \text{Mor } \mathbb{C}$. Denote $\Sigma := \bigcup_{i \geq 1} \Sigma^i$.

Recall that the bar-cobar resolution ΩBC (e.g. [11], where the noncoloured case is treated - but the coloured case is completely analogous) is a quasi-free V -operad generated by V - Σ -module $k\langle \Sigma \rangle$, where the degree of $\sigma \in \Sigma^n$ is $n-1$. The derivation differential is given by

$$\begin{aligned} \partial(\overleftarrow{f_n} \dots \overleftarrow{f_1}) &:= \sum_{i=1}^{n-1} (-1)^{i+n+1} (\overleftarrow{f_n} \dots \overleftarrow{f_{i+1}}) \circ (\overleftarrow{f_i} \dots \overleftarrow{f_1}) + \\ &\quad + \sum_{i=1}^{n-1} (-1)^{n-i} (\overleftarrow{f_n} \dots \overleftarrow{f_{i+1}f_i} \dots \overleftarrow{f_1}). \end{aligned}$$

The projection $\phi_{\mathcal{C}} : \Omega BC \rightarrow \Omega B^1 \mathcal{C} \cong \mathcal{C}$ onto the sub V - Σ -module of weight 1 elements is a quism.

5.1 Theorem. Let $\llbracket - \rrbracket_2^{\text{NS}} : \Omega BC \rightarrow \Omega BC \otimes \Omega BC$ be a linear map satisfying $\llbracket a \circ b \rrbracket_2^{\text{NS}} =$

$\llbracket a \rrbracket_2^{\text{NS}} \circ \llbracket b \rrbracket_2^{\text{NS}}$ for all $a, b \in \Omega\text{BC}$ and determined by its values on generators:

$$\begin{aligned} & \llbracket (\overleftarrow{f_n} \dots \overleftarrow{f_1}) \rrbracket_2^{\text{NS}} := \\ & (\overleftarrow{f_n} \dots \overleftarrow{f_1}) \otimes (\overleftarrow{f_n \dots f_1}) + \\ & + \sum_{\substack{1 \leq m \leq n-1 \\ 1 \leq j_1 < \dots < j_m \leq n-1}} (-1)^\epsilon (\overleftarrow{f_n} \dots \overleftarrow{f_{j_m+1}}) \dots (\overleftarrow{f_{j_1}} \dots \overleftarrow{f_1}) \otimes (\overleftarrow{f_n \dots f_{j_m+1}} \dots \overleftarrow{f_{j_1} \dots f_1}), \end{aligned}$$

where $\epsilon := mn + \frac{1}{2}m(m-1) + \sum_{i=1}^k j_i$. Then $\llbracket - \rrbracket_2^{\text{NS}}$ induces, via (9) and (8), the maps $\llbracket - \rrbracket_n^{\text{NS}}$ and $\llbracket - \rrbracket_n$ of Lemma 3.6. Moreover,

$$(1^{\otimes i} \otimes \llbracket - \rrbracket_a^{\text{NS}} \otimes 1^{\otimes b-i-1}) \llbracket - \rrbracket_b^{\text{NS}} = \llbracket - \rrbracket_{a+b-1}^{\text{NS}}.$$

Proof. We apply Lemma 3.9. The only nontrivial properties to verify are $\partial \llbracket - \rrbracket_2^{\text{NS}} = \llbracket \partial - \rrbracket_2^{\text{NS}}$ and $(\llbracket - \rrbracket_2^{\text{NS}} \otimes 1) \llbracket - \rrbracket_2^{\text{NS}} = (1 \otimes \llbracket - \rrbracket_2^{\text{NS}}) \llbracket - \rrbracket_2^{\text{NS}}$. This can be done directly, but it is annoying and doesn't explain the origin of $\llbracket - \rrbracket_2^{\text{NS}}$. Thus we go another way. There is the following description of ΩBC . Let

$$C_*(I) := k\langle (\mathbf{0}), (\mathbf{1}), (\mathbf{01}) \rangle$$

be the simplicial chain complex of the interval, i.e. $|(\mathbf{0})| = |(\mathbf{1})| = 0$, $|(\mathbf{01})| = 1$ and $\partial(\mathbf{0}) = \partial(\mathbf{1}) = 0$, $\partial(\mathbf{01}) = (\mathbf{1}) - (\mathbf{0})$. Then

$$\Omega\text{BC} = \frac{\bigoplus_{n \geq 0} \mathcal{C}^{\circ(n+1)} \otimes C_*(I)^{\otimes n}}{M}, \quad (29)$$

where the subspace M is spanned by

$$\begin{aligned} & f_n \otimes \dots \otimes f_{i+1} \otimes f_i \otimes \dots \otimes f_1 \otimes c_{n-1} \otimes \dots \otimes c_{i+1} \otimes (\mathbf{0}) \otimes c_{i-1} \otimes \dots \otimes c_1 + \\ & - f_n \otimes \dots \otimes f_{i+1} f_i \otimes \dots \otimes f_1 \otimes c_{n-1} \otimes \dots \otimes c_{i+1} \otimes c_{i-1} \otimes \dots \otimes c_1 \end{aligned}$$

for any $f_n, \dots, f_1 \in \mathcal{C}$ of right colours and any $c_{n-1}, \dots, c_1 \in C_*(I)$. Let the grading and the differential ∂ on ΩBC be induced by $C_*(I)$ (\mathcal{C} is concentrated in degree 0) in the standard way. The operadic composition is defined by

$$\begin{aligned} & (f_n \otimes \dots \otimes f_1 \otimes c_{n-1} \otimes \dots \otimes c_1) \circ (g_m \otimes \dots \otimes g_1 \otimes d_{m-1} \otimes \dots \otimes d_1) := \\ & (f_n \otimes \dots \otimes f_1 \otimes g_m \otimes \dots \otimes g_1 \otimes c_{n-1} \otimes \dots \otimes c_1 \otimes d_{m-1} \otimes \dots \otimes d_1). \end{aligned}$$

A dg V -operad isomorphism with the previous description is easily seen to be

$$f_n \otimes \dots \otimes f_1 \otimes c_{n-1} \otimes \dots \otimes c_1 \mapsto (\overleftarrow{f_n} \dots \overleftarrow{f_{j_m+1}}) \dots (\overleftarrow{f_{j_1}} \dots \overleftarrow{f_1}), \quad (30)$$

where $c_{j_m} = c_{j_{m-1}} = \dots = c_{j_1} = (\mathbf{1})$ and all other c_i 's equal $(\mathbf{01})$ (remember we can get rid of $(\mathbf{0})$ using the defining relations). The point is that $C_*(I)$ carries the obvious coassociative coproduct

$$\Delta(\mathbf{0}) = (\mathbf{0}) \otimes (\mathbf{0}), \quad \Delta(\mathbf{1}) = (\mathbf{1}) \otimes (\mathbf{1}), \quad \Delta(\mathbf{01}) = (\mathbf{0}) \otimes (\mathbf{01}) + (\mathbf{01}) \otimes (\mathbf{1})$$

and there is also the trivial coproduct on \mathcal{C} given by $\Delta(c) = c \otimes c$. These induce coproduct on ΩBC by

$$\Delta(f_n \otimes \dots \otimes f_1 \otimes c_{n-1} \otimes \dots \otimes c_1) := \left(\underbrace{(\Delta \otimes \dots \otimes \Delta)}_{2n-1 \text{ times}} (f_n \otimes \dots \otimes f_1 \otimes c_{n-1} \otimes \dots \otimes c_1) \right) \cdot \tau,$$

where $\tau \in \Sigma_{4n-2}$ rearranges the factors in the expected way, which will be obvious from the following computation. Denote $c^0 := (\mathbf{0}) \otimes (\mathbf{01})$, $c^1 := (\mathbf{01}) \otimes (\mathbf{1})$ and for the rest of the proof, let's order the factor of the tensor products from right to left, i.e. $(\mathbf{01})$ is in position 2 in c^1 and $(\mathbf{1})$ is in position 1. Then

$$\begin{aligned} & \Delta(f_n \otimes \cdots \otimes f_1 \otimes \underbrace{(\mathbf{01}) \otimes \cdots \otimes (\mathbf{01})}_{n-1 \text{ times}}) = \\ &= \sum_{\substack{0 \leq m \leq n-1 \\ 1 \leq j_1 < \cdots < j_m \leq n-1}} \left(f_n \otimes f_n \otimes \cdots \otimes f_1 \otimes f_1 \otimes c^0 \otimes \cdots \otimes \underbrace{c^1}_{\text{position } j_m} \otimes \cdots \otimes \underbrace{c^1}_{\text{position } j_1} \otimes \cdots \otimes c^0 \right) \cdot \tau, \end{aligned}$$

where c^1 appears only at positions j_1, \dots, j_m . Applying τ and multiplying yields

$$\begin{aligned} & \sum_{\substack{0 \leq m \leq n-1 \\ 1 \leq j_1 < \cdots < j_m \leq n-1}} (-1)^\epsilon \left(f_n \otimes \cdots \otimes f_1 \otimes (\mathbf{0}) \otimes \cdots \otimes \underbrace{(\mathbf{01})}_{j_m} \otimes \cdots \otimes \underbrace{(\mathbf{01})}_{j_1} \otimes \cdots \otimes (\mathbf{0}) \right) \otimes \\ & \otimes \left(f_n \otimes \cdots \otimes f_1 \otimes (\mathbf{01}) \otimes \cdots \otimes \underbrace{(\mathbf{1})}_{j_m} \otimes \cdots \otimes \underbrace{(\mathbf{1})}_{j_1} \otimes \cdots \otimes (\mathbf{01}) \right), \end{aligned}$$

where $\epsilon = mn + \frac{1}{2}m(m-1) + \sum_{i=1}^m j_i$ comes from the Koszul convention. This is exactly the claimed formula under the isomorphisms (30).

It is easily seen that $\Delta(a \circ b) = \Delta(a) \circ \Delta(b)$. Δ is the coproduct induced on the quotient (29) by the tensor product of coassociative dg coalgebras $C_*(I)$ and \mathcal{C} . It is a standard fact that the tensor product is also a coassociative dg coalgebra, hence $\partial\Delta = \Delta\partial$, $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$. Then $\llbracket - \rrbracket_2^{\text{NS}} := \Delta$ has the properties (C2)–(C6).

Originally, we found the coproduct of this lemma by hand. We are indebted to Benoit Fresse for suggesting its origin in $C_*(I)$. \square

A completely explicit cofibrant resolution \mathcal{D}_∞ of $\mathcal{D} = \mathcal{A}_{\mathcal{C}}$ gives rise to a cohomology theory for $\mathcal{A}_{\mathcal{C}}$ -algebras (i.e. \mathcal{C} -shaped diagrams of \mathcal{A} -algebras) describing their deformations. This is explained in [9]. Unfortunately, the description of ∂ on \mathcal{D}_∞ given in Theorem 3.15 is not even explicit enough to write down the codifferential δ on the corresponding deformation complex $\text{Der}^*(\mathcal{D}_\infty, \mathcal{E}nd_W)$, not to mention the rest of the L_∞ -structure. For the basic example $\mathcal{A} = \mathcal{A}ss$, we already proved in [1] that $(\text{Der}^*(\mathcal{D}_\infty, \mathcal{E}nd_W), \delta)$ is isomorphic to the Gerstenhaber-Schack complex (see [4]) $(C_{\text{GS}}^*(D, D), \delta_{\text{GS}})$ (of a diagram D) for *some* resolution \mathcal{D}_∞ . The method, however, doesn't allow to find \mathcal{D}_∞ explicitly. We conjecture that this \mathcal{D}_∞ has the form given by Theorem 3.15:

5.2 Conjecture. In Theorem 3.15, let $\mathcal{A} := \mathcal{A}ss$, let $\mathcal{A}_\infty := \mathcal{A}ss_\infty$ be the minimal resolution of $\mathcal{A}ss$ and let $\mathcal{C}_\infty = \Omega BC$. Then there are $\omega(x, f)$'s such that

$$(\text{Der}^*(\mathcal{D}_\infty, \mathcal{E}nd_W), \delta) \cong (C_{\text{GS}}^*(D, D), \delta_{\text{GS}}).$$

Another very interesting problem is to find an operadic interpretation of Cohomology Comparison Theorem: Recall that CCT, proved in [4], is a theorem relating

deformations of the diagram of associative algebras to deformations of a single associative algebra. The point is that the deformations of the single algebra are described by Hochschild complex equipped with a dg Lie algebra structure given by Hochschild differential and Gerstenhaber bracket. On the other hand, in known examples (see [3]), the L_∞ -structure on operadic deformation complex of the diagram has nontrivial higher brackets (see [3]). This suggest that this L_∞ -algebra can be rectified to the one given by CCT.

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